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Ground state sign-changing solutions for a class of generalized quasilinear Schrödinger equations with a Kirchhoff-type perturbation

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Abstract. We investigate the existence of ground state sign-changing solutions for the following elliptic equation of Kirchhoff type:

$$\left(1+b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 \mathrm{d}x\right)\left[-\mathrm{div}(g^2(u)\nabla u)+g(u)g'(u)|\nabla u|^2\right] + V(x)u = K(x)f(u),$$

where $x \in \mathbb{R}^3, b > 0, g \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^+), V(x)$, and $K(x)$ are both positive continuous functions. First, using some new analytical techniques and non-Nehari manifold method, we obtain one ground state sign-changing solution $v_b = G^{-1}(u_b)$. Moreover, we prove that the energy of $v_b = G^{-1}(u_b)$ is strictly larger than twice that of the ground state solutions of Nehari type. We also establish the convergence property of $v_b = G^{-1}(u_b)$ as the parameter $b \searrow 0$. Our results improve and generalize some results in Li et al. (J Math Anal Appl 443:11–38, 2016).

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Keywords. Generalized quasilinear Schrödinger equations, Kirchhoff-type perturbation, ground state sign-changing solutions, non-Nehari manifold method.

1. Introduction

In this paper, we are concerned with a class of non-linear elliptic equations:

$$\left(1+b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 \mathrm{d}x\right)\left[-\mathrm{div}(g^2(u)\nabla u)+g(u)g'(u)|\nabla u|^2\right] + V(x)u = K(x)f(u), \tag{1.1}$$

where $x \in \mathbb{R}^3, b > 0, V(x)$, and $K(x)$ are both positive functions; f is a continuous function and $g \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^+)$.

Birkhäuser

In recent years, there has been increasing attention to the following generalized quasilinear Schrödinger equations:

$$- \operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x,u) \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $N \geq 3$, see, such as [7,8,15–18,37–40] and the references therein. Methematically, it is, indeed, a hot issue in non-linear analysis to study the existence of solitary wave solutions for the following quasi-linear Schrödinger equation:

$$i\partial_t z = -\Delta z + W(x)z - k(x,|z|) - \Delta l(|z|^2)l'(|z|^2)z, \tag{1.3}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, and $l : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are suitable functions. For various types of l, the quasilinear equation of the form (1.1) has been derived from models of several physical phenomenon. In particular, when $l(s) = s$, Eq. (1.1) was used to derive the superfluid film [20,22] equation in fluid mechanics by Kurihara [20]. For more physical background, please refer to literatures [5,6,9,19,23,33,35,36]. In fact, (1.3) with $l(t) = t^\alpha$ for some $\alpha \geq 1$, see [24–26,32] and the references therein. However, in our mind, only in the recent papers [16,40], Eq. (1.3) with a general l has been considered.

When $b = 0$, then (1.1) is reduced to the following equation:

$$- \operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = K(x)f(u), \qquad x \in \mathbb{R}^3. \tag{1.4}$$

If $g^2(u) = 1 + 2u^2$ in (1.4), then (1.4) derives the following equation:

$$- \Delta u + V(x)u - \Delta(u^2)u = K(x)f(u), \quad x \in \mathbb{R}^3. \tag{1.5}$$

Moreover, if we choose $g(t) = 1$ in (1.1), then (1.1) reduces to the following equation:

$$- \left(1 + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right) \Delta u + V(x)u = f(x,u), \qquad x \in \mathbb{R}^3. \tag{1.6}$$

As far as we know, this problem is related to the stationary analog of the evolution equation of Kirchhoff type:

$$u_{tt} - \left(1 + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right) \triangle u + V(x)u = f(x,u), \tag{1.7}$$

where ∇u denotes the spatial gradient of u, which was proposed by Kirchhoff [21] as an extension of the classical D'Alemberts wave equation for free vibrations of elastic strings. Equation (1.6) appears in many fields, such as physical, engineering and other sciences, and in those situations, model (1.6) is considered to be a good approximation for describing non-linear vibrations of beams or plates. There has been a great deal of attention devoted to the existence and multiplicity of solutions for (1.6). For related work, we can refer to [27,29,48–50] and so on.

Note that (1.2) is more general than (1.5). Therefore, it is more necessary to study (1.2). For related work, we can refer to [14–18,30,37–40]. Based on these work, in very recent years (1.2), derives two different kinds of equations. One is generalized quasilinear Schrödinger–Maxwell system, which has been considered in [13,51]. The other is (1.1), which is considered to be a generalization of (1.2). Therefore, it is very important for us to study (1.1).

To the authors' knowledge, there are few papers on the existence of the ground state solution and sign-changing solution for (1.1) except for Li et al. [28]. They called Eq. (1.1) as a Kirchhoff-type perturbation of the generalized quasilinear Schrödinger equation (1.2) due to the appearance of the non-local term $\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 \mathrm{d}x$. After the work of [28], there are no papers concerned with the existence of the ground state sign-changing solutions with general non-linearity. In this paper, we will deal with the existence of the ground state solution sign-changing solution to Eq. (1.2) with general non-linearity. What is more, we establish an interesting result that the same conclusion can be also derived for Eq. (1.1), which adds a perturbation in (1.2).

To guarantee the compactness, as in [10,42], we assume that $g, V(x)$ and $K(x)$ satisfy

(g) $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^+)$ is even with $g'(t) \geq 0$ for all $t \in \mathbb{R}^+$ and $g(0) = 1$.

(V) $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}), V(x) > 0$ for all $x \in \mathbb{R}^3$ and $H \subset H^1(\mathbb{R}^3)$, such that, for $2 < q < 6$, the embedding

$$H \hookrightarrow L^q(\mathbb{R}^3)$$

is compact, where

$$H := \begin{cases} H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\}, & \text{if } V(x) \text{ is a constant,} \\ \left\{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) \mathrm{d}x < \infty\right\}, & \text{if } V(x) \text{ is not a constant,} \end{cases} \tag{1.8}$$

with the following norm:

$$\|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) \mathrm{d}x\right)^{\frac{1}{2}}.$$

Clearly,

$$H^1(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\right\},$$

with the norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \mathrm{d}x\right)^{\frac{1}{2}}.$$

(K) $K \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3, \mathbb{R})$ and $K(x) > 0$ for all $x \in \mathbb{R}^3$.

To establish the existence of ground state sign-changing solutions, we need to make the following assumptions:

(f_1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and

$$\lim_{t \to 0} \frac{f(t)}{g(t)G(t)} = 0,$$

where $G(t) = \int_0^t g(s) \mathrm{d}s$ for all $t \in \mathbb{R}$.

(f_2) f has a "quasicritical" growth, namely $\lim_{|t| \to \infty} \frac{f(t)}{g(t)G(t)^5} = 0$.

(f_3) $\lim_{|t| \to \infty} \frac{f(t)}{g(t)G(t)^3} = \infty$.

In fact, by applying the constraint variational method and quantitative deformation lemma, Li et al. [28] proved the existence of ground state solutions and sign-changing solutions to (1.1) when $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ satisfies (f_1)–(f_3) and (f_4') $f/(g|G^3|)$ is increasing on $(-\infty, 0)$ and $(0, \infty)$, respectively, and $\lim_{|t| \to \infty} F(t)/G^4(t) = \infty$, where $F(t) = \int_0^t f(s) \mathrm{d}s$ for all $t \in \mathbb{R}$.

In addition, they also assume that $(V, K) \in \mathcal{K}$, that is, V and K satisfy:

(V') the potential function V is positive on $\mathbb{R}^3$ and belongs to $L^\infty(\mathbb{R}^3) \cap \mathcal{C}^\alpha(\mathbb{R}^3)$ for some $\alpha \in (0,1)$;

(K') $K \in L^\infty(\mathbb{R}^3) \cap \mathcal{C}^\alpha(\mathbb{R}^3)$ is positive;

(K_1) If $\{S_n\} \subset \mathbb{R}^3$ is a sequence of Borel sets, such that $|S_n| \leq M$, for all n and some $M > 0$, then we have

$$\lim_{R \to +\infty} \int_{S_n \setminus B_R} K(x) \mathrm{d}x = 0, \quad \text{uniformly in } n \in \mathbb{N},$$

where $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$;

(K_2) For some $p \in [2, 6)$ and $A_p \in \mathbb{R}^+$, it holds that

$$\limsup_{|x| \to \infty} \frac{K(x)}{V(x)^{(6-p)/4}} = [1 - \mathrm{sgn}(p-2)] A_p,$$

where $\mathrm{sgn}(p-2) = 0$ if $p = 2$ and $\mathrm{sgn}(p-2) = 1$ if $p > 2$.

With these hypotheses, the space E given by

$$E = \left\{ u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x) u^2 \mathrm{d}x < \infty \right\}$$

with the same norm as in H is compactly embedded into the weighted Lebesgue space:

$$L_K^q(\mathbb{R}^3) = \left\{ u : u \,\text{is measurable on}\, \mathbb{R}^3 \,\text{and} \int_{\mathbb{R}^3} K(x) |u|^q \mathrm{d}x < \infty \right\},$$

for some $q \in (2, 6)$, see ([1], Proposition 2.1).

Obviously, (V') implies that the potential V may vanish at infinity, and then, Eq. (1.1) becomes a zero mass problem. Therefore, the main difficulty lies in the lack of compactness. To overcome this difficulty, some scholars suppose that $(K'), (K_1)$, and (K_2) hold, such as [1]. However, in this paper, we can skillfully get a compact embedding via the conditions (V) and (K).

Since the term $\int_{\mathbb{R}^3} g^2(u) |\nabla u|^2 \mathrm{d}x$ is not well defined in H, it is necessary to propose a variable substitution as follows, in detail. For any $v \in H$, as [40], we make a change of variable as

$$u = G^{-1}(v) \quad \text{and} \quad G(u) = \int_0^u g(t) \mathrm{d}t,$$

then

$$\int_{\mathbb{R}^3} g^2(u) |\nabla u|^2 \mathrm{d}x = \int_{\mathbb{R}^3} g^2(G^{-1}(v)) |\nabla G^{-1}(v)|^2 \mathrm{d}x := |\nabla v|_2^2 < \infty.$$

It is easy to see that a function $u : \mathbb{R}^3 \to \mathbb{R}$ is called a weak solution of (1.1), if $v \in H$ and for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, it holds

$$\begin{aligned} &(1 + b|\nabla v|_2^2) \int_{\mathbb{R}^3} \nabla v \cdot \nabla \phi \mathrm{d}x + \int_{\mathbb{R}^3} V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \phi \mathrm{d}x \\ &\quad = \int_{\mathbb{R}^3} K(x) \frac{f(G^{-1}(v))}{g(G^{-1}(v))} \phi \mathrm{d}x. \end{aligned}$$

Define energy functional as

$$\mathcal{I}_b(v) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} V(x) G^{-1}(v)^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} K(x) F(G^{-1}(v))\mathrm{d}x, \tag{1.9}$$

where $v \in H$ and $F(u) = \int_0^u f(s)ds$. Hence, the critical point of $\mathcal{I}'$ is solutions of (1.1). The functional $\mathcal{I}_b$ is well defined for every $v \in H$ and $\mathcal{I}_b \in \mathcal{C}^1(H, \mathbb{R})$. Moreover, if v is a critical point for $\mathcal{I}_b'$, then for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we have

$$\langle \mathcal{I}_b'(v), \psi\rangle = (1 + b|\nabla v|_2^2)\int_{\mathbb{R}^3} \nabla v \nabla \psi \mathrm{d}x + \int_{\mathbb{R}^3} V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\psi - \int_{\mathbb{R}^3} K(x)\frac{f(G^{-1}(v))}{g(G^{-1}(v))}\psi \mathrm{d}x. \tag{1.10}$$

Thus, to obtain the solutions of (1.1), it suffices to study the existence of solutions of the following equation:

$$-\left(1 + b\int_{\mathbb{R}^3} |\nabla v|^2 \mathrm{d}x\right)\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = K(x)\frac{f(G^{-1}(v))}{g(G^{-1}(v))}, \quad x \in \mathbb{R}^3. \tag{1.11}$$

For convenience, we rewrite Eq. (1.9) in the following form:

$$-\left(1 + b\int_{\mathbb{R}^3} |\nabla v|^2 \mathrm{d}x\right)\Delta v + V(x)v = K(x)\tilde{f}(x, v), \quad x \in \mathbb{R}^3, \tag{1.12}$$

and the corresponding energy functional is

$$\mathcal{J}_b(v) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla v|^2 + V(x)v^2)\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} K(x)\tilde{F}(x, v)\mathrm{d}x,$$

where

$$\tilde{f}(x, v) = \frac{f(G^{-1}(v))}{g(G^{-1}(v))} + \frac{V(x)}{K(x)}v - \frac{V(x)}{K(x)}\frac{G^{-1}(v)}{g(G^{-1}(v))}$$

and

$$\tilde{F}(x, v) = \int_0^v \tilde{f}(x, s)\mathrm{d}s = F(G^{-1}(v)) + \frac{1}{2}\frac{V(x)}{K(x)}v^2 - \frac{V(x)}{K(x)}|G^{-1}(v)|^2.$$

If $b = 0$, then (1.12) will reduce to the following equation:

$$-\Delta v + V(x)v = K(x)\tilde{f}(x, v), \quad x \in \mathbb{R}^3. \tag{1.13}$$

Clearly, if v is a critical point of $\mathcal{J}_b(v)$, then $u = G(v)$ is a weak solution of (1.1). In addition, by the monotonicity of G in Lemma 2.1 below, if v is a ground state sign-changing solution of (1.12), then $u = G(v)$ is a ground state sign-changing solution for (1.1). Thus, we just need to study Problem (1.12).

For the sake of achieving our results, we also need to make the following assumption:

($\tilde{f}_4$) There exists a $\theta_0 \in (0,1)$, such that for any $x \in \mathbb{R}^3, t > 0$ and $\tau \neq 0$

$$K(x)\left[\frac{\tilde{f}(x,\tau)}{\tau^3} - \frac{\tilde{f}(x,t\tau)}{(t\tau)^3}\right]\operatorname{sgn}(1-t) + \theta_0 V(x)\frac{|1-t^2|}{(t\tau)^2} \geq 0.$$

It is important to highlight that ($\tilde{f}_4$) plays a key role in establishing the existence of the ground state sign-changing solutions. What is more, ($\tilde{f}_4$) is much weaker than the following condition:

($\tilde{f}_4'$) $\frac{\tilde{f}(x,t)}{|t|^3}$ is non-decreasing on $\mathbb{R}\backslash\{0\}$.

In fact, many functions can satisfy assumption ($\tilde{f}_4$), but not ($\tilde{f}_4'$). Motivated by [10], we give the following example to illustrate this point: let $g(t) = 1, V \equiv 1$, and $0 < K(x) \leq M$ for all $x \in \mathbb{R}^3$:

$$f(x,t) = \begin{cases} |t|^3 t, & \text{if } |t| \leq \varrho, \\ \alpha |t|^3 t + \frac{1}{3M}t, & \text{if } |t| > \varrho, \end{cases}$$

$\alpha, \varrho > 0$, and $3(1-\alpha)\varrho^3 M = 1$. Obviously, $\tilde{f} = f$ satisfies ($\tilde{f}_4$) with $\theta_0 = 1/2$ but does not satisfy ($\tilde{f}_4'$).

Furthermore, if $v \in H$ is a solution of (1.1) and $v^{\pm} \neq 0$, then v is a sign-changing solution of (1.7), where

$$v^+(x) := \max\{v(x), 0\} \quad \text{and} \quad v^-(x) := \min\{v(x), 0\}.$$

As we all know, various ways have been adopted to prove the existence of sign-changing solutions for elliptic equation, such as by constructing invariant sets and descending flow in [2], via the Ekeland's variational principle and the implicit function theorem in [34], applying variational method together with the Brouwer degree theory in [3], by Morse index theory in [11], and using diagonal principle with non-Nehari manifold method in [3,10,13,14,43–46].

Next, we give an essential decomposition which is useful in these methods to seek sign-changing solutions for (1.1), for any $v \in H$:

$$\mathcal{J}'_0(v) = \mathcal{J}'_0(v^+) + \mathcal{J}'_0(v^-), \tag{1.14}$$

$$\langle \mathcal{J}'_0(v), v^+\rangle = \langle \mathcal{J}'_0(v^+), v^+\rangle, \quad \langle \mathcal{J}'_0(v), v^-\rangle = \langle \mathcal{J}'_0(v^-), v^-\rangle, \tag{1.15}$$

where $\mathcal{J}_0 : H \to \mathbb{R}$ is the energy functional of (1.4) given by the following:

$$\mathcal{J}_0(v) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla v|^2 + V(x)v^2)\mathrm{d}x - \int_{\mathbb{R}^3} K(x)\tilde{F}(x,v)\mathrm{d}x$$

and

$$\langle \mathcal{J}'_0(v), \varphi\rangle = \int_{\mathbb{R}^3}(\nabla v \nabla \varphi + V(x)v\varphi)\mathrm{d}x - \int_{\mathbb{R}^3} K(x)\tilde{f}(x,v)\varphi\mathrm{d}x.$$

For the functional $\mathcal{J}_b$, we have

$$\mathcal{J}_b(v) = \mathcal{J}_b(v^+) + \mathcal{J}_b(v^-) + \frac{b}{2}\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2, \tag{1.16}$$

$$\langle \mathcal{J}'_b(v), v^+\rangle = \langle \mathcal{J}'_b(v^+), v^+\rangle + b\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2, \tag{1.17}$$

$$\langle \mathcal{J}'_b(v), v^- \rangle = \langle \mathcal{J}'_b(v^-), v^- \rangle + b\|\nabla v^+\|_2^2 \|\nabla v^-\|_2^2. \tag{1.18}$$

Motivated by the above-mentioned works, we will consider the following minimization problems:

$$m_b := \inf_{v \in \mathcal{M}_b} \mathcal{J}_b(v) \quad \text{and} \quad m_0 := \inf_{v \in \mathcal{M}_0} \mathcal{J}_0(v), \tag{1.19}$$

where

$$\mathcal{M}_b := \left\{ v \in H : v^\pm \neq 0, \langle \mathcal{J}'_b(v), v^+ \rangle = \langle \mathcal{J}'_b(v), v^- \rangle = 0 \right\} \tag{1.20}$$

and

$$\mathcal{M}_0 := \left\{ v \in H : v^\pm \neq 0, \langle \mathcal{J}'_0(v), v^+ \rangle = \langle \mathcal{J}'_0(v), v^- \rangle = 0 \right\}, \tag{1.21}$$

the minimizers correspond to the sign-changing solutions for Problems (1.1) and (1.2), respectively.

In the present paper, we intend to prove that the energy of any sign-changing solutions of (1.1) is larger than twice that of the ground state solutions of (1.1), and establish the convergence property of a least energy sign-changing solution for Problem (1.1) as $b \searrow 0$. Define the following Nehari manifolds:

$$\mathcal{N}_b := \{ v \in H : v \neq 0, \langle \mathcal{J}'_b(v), v \rangle = 0 \rangle \} \tag{1.22}$$

and

$$\mathcal{N}_0 := \{ v \in H : v \neq 0, \langle \mathcal{J}'_0(v), v \rangle = 0 \rangle \} \tag{1.23}$$

with

$$c_b := \inf_{v \in \mathcal{N}_b} \mathcal{J}_b(v) \quad \text{and} \quad c_0 := \inf_{v \in \mathcal{N}_0} \mathcal{J}_0(v), \tag{1.24}$$

which play an active role to seek the ground state solutions of Nehari type for (1.1) and (1.4).

Now, we state our main results by the following theorems.

Theorem 1.1. *Suppose that* $(g), (V), (K)$, *(f_1)–(f_3), and $(\tilde{f}_4)$ are satisfied. Then, problem* (1.12) *has a sign-changing solution $v_b \in \mathcal{M}_b$, such that $\mathcal{J}_b(v_b) = \inf_{\mathcal{M}_b} \mathcal{J}_b > 0$, which has precisely two nodal domains.*

Theorem 1.2. *Suppose that* $(g), (V), (K)$, *(f_1)–(f_3), and $(\tilde{f}_4)$ are satisfied. Then, problem* (1.12) *has a solution $\bar{v} \in \mathcal{N}_b$, such that $\mathcal{J}_b(\bar{v}) = \inf_{\mathcal{N}_b} \mathcal{J}_b$, moreover, $m_b > 2c_b$.*

Theorem 1.3. *Suppose that* $(g), (V), (K)$, *(f_1)–(f_3), and $(\tilde{f}_4)$ are satisfied. Then, problem* (1.13) *has a sign-changing solution $w_0 \in \mathcal{M}_0$, such that $\mathcal{J}_0(w_0) = \inf_{\mathcal{M}_0} \mathcal{J}_0 > 0$, which has precisely two nodal domains. Furthermore, for any sequence $\{b_n\}$ with $b_n \searrow 0$ as $n \to \infty$, there exists a subsequence which we label in the same way, such that $v_{b_n} \to v_0$ in H, where $v_0 \in \mathcal{M}_0$ is a sign-changing solution of* (1.13) *with $\mathcal{J}_0(v_0) = \inf_{\mathcal{M}_0} \mathcal{J}_0 > 0$.*

Remark 1.4. (I) In this paper, the problem (1.1) possesses the non-local term $(\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 \mathrm{d}x)\Delta u$, as we mentioned above, $\mathcal{J}_b$ no longer has the properties (1.14) and (1.15), and it is rather difficult to show that $\mathcal{M}_b \neq \emptyset$. To seek sign-changing solutions, we introduce a condition $(\tilde{f}_4)$ much weaker than $(\tilde{f}'_4)$. With $(\tilde{f}_4)$, we use a new ideas, i.e., non-Nehari manifold method to prove the existence of ground state sign-changing solutions for (1.1).

(II) Note that $f \in \mathcal{C}^1(\mathbb{R}^3)$ is a key point in seeking ground state energy sign-changing solutions in [28]. However, in this paper, we only assume that $f \in \mathcal{C}(\mathbb{R}^3)$.

The paper is organized as follows. In Sect. 2, some preliminary lemmas are presented. In Sect. 3, we prove a solution of (1.1) with two nodal domains using critical point obtained in Sect. 2 as a component. Sections 4 and 5 are devoted to the proof of Theorems 1.2 and 1.3, respectively.

2. Some preliminary lemmas

In this section, we present some fundamental lemmas and corollaries. Now, let us review the following lemma which has been proved in [30].

Lemma 2.1 [30]. *For the functions g, G, and G^{-1}, the following properties hold:*

1. *the functions $G(\cdot)$ and $G^{-1}(\cdot)$ are strictly increasing and odd.*
2. *$G(s) \leq g(s)s$ for all $s \geq 0$; $G(s) \geq g(s)s$ for all $s \leq 0$.*
3. *$g(G^{-1}(s)) \geq g(0) = 1$ for all $s \in \mathbb{R}$.*
4. *$\frac{G^{-1}(s)}{s}$ is decreasing on $(0, +\infty)$ and increasing on $(-\infty, 0)$.*
5. *$|G^{-1}(s)| \leq \frac{1}{g(0)}|s| = |s|$ for all $s \in \mathbb{R}$.*
6. *$\frac{|G^{-1}(s)|}{g(G^{-1}(s))} \leq \frac{1}{g^2(0)}|s| = |s|$ for all $s \in \mathbb{R}$.*
7. *$\frac{G^{-1}(s)s}{g(G^{-1}(s))} \leq |G^{-1}(s)|^2$ for all $s \in \mathbb{R}$.*
8. *$\lim_{|s|\to 0} \frac{G^{-1}(s)}{s} = \frac{1}{g(0)} = 1$ and*

$$\lim_{|s|\to+\infty} \frac{G^{-1}(s)}{s} = \begin{cases} \frac{1}{g(\infty)}, & \text{if } g \text{ is bounded}, \\ 0, & \text{if } g \text{ is unbounded}. \end{cases}$$

Lemma 2.2. *Assume that (g) and (f_1)–(f_3) hold. Then, the function $\tilde{f}(x, s)$ has the following properties:*

$(\tilde{f_1})$ *$\tilde{f} \in \mathcal{C}(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $\lim_{s\to 0} \frac{\tilde{f}(x,s)}{s} = 0$;*
$(\tilde{f_2})$ *$\tilde{f}$ has a "quasicritical" growth, that is,*

$$\lim_{|s|\to\infty} \frac{\tilde{f}(x, s)}{s^5} = 0.$$

$(\tilde{f_3})$ *$\tilde{f}$ is superquadratic at infinity, that is,*

$$\lim_{|s|\to\infty} \frac{\tilde{f}(x, s)}{s^3} = \infty.$$

Proof. Using (8) in Lemma 2.1, one has

$$\begin{aligned}\lim_{s\to 0}\frac{\tilde{f}(x,s)}{s} &= \lim_{s\to 0}\frac{f(G^{-1}(s))}{sg(G^{-1}(s))}+\frac{V(x)}{K(x)}\frac{1}{2}\lim_{s\to 0}\left(1-\frac{G^{-1}(s)}{sg(G^{-1}(s))}\right)\\ &= 0+\frac{V(x)}{K(x)}\frac{1}{2}\lim_{s\to 0}\left(1-\frac{1}{g(0)}\right)\\ &= 0.\end{aligned}$$

Since $\tilde{f}\in\mathcal{C}(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$, then $(\tilde{f}_1)$ holds. Moreover, we have

$$\lim_{|s|\to\infty}\frac{\tilde{f}(x,s)}{s^5}=\lim_{|s|\to\infty}\frac{1}{s^5}\frac{f(G^{-1}(s))}{g(G^{-1}(s))}+\frac{V(x)}{K(x)}\frac{1}{2}\lim_{|s|\to\infty}\left(\frac{1}{s^4}-\frac{1}{s^5}\frac{G^{-1}(s)}{g(G^{-1}(s))}\right).$$

Let $t=G^{-1}(s)$, then the above equality can deduce that

$$\begin{aligned}\lim_{|s|\to\infty}\frac{\tilde{f}(x,s)}{s^5} &= \lim_{|t|\to\infty}\frac{f(t)}{g(t)G(t)^5}+\frac{V(x)}{K(x)}\frac{1}{2}\lim_{|s|\to\infty}\left(\frac{1}{s^4}-\frac{1}{s^5}\frac{G^{-1}(s)}{g(G^{-1}(s))}\right)\\ &= 0+\frac{V(x)}{K(x)}\frac{1}{2}\lim_{|s|\to\infty}\left(\frac{1}{s^4}-\frac{1}{s^4}\frac{1G^{-1}(s)}{sg(G^{-1}(s))}\right)=0,\end{aligned}$$

then $(\tilde{f}_2)$ holds. Next, by (f_3), we have

$$\lim_{|s|\to\infty}\frac{\tilde{f}(x,s)}{s^3}=\lim_{|t|\to\infty}\frac{f(t)}{g(t)G^3(t)}+\frac{V(x)}{K(x)}\frac{1}{2}\lim_{|s|\to\infty}\left(\frac{1}{s^2}-\frac{1}{s^2}\frac{G^{-1}(s)}{sg(G^{-1}(s))}\right)=\infty,$$

and thus, $(\tilde{f}_3)$ holds. □

Lemma 2.3. *Suppose that* $(V),(K)$, *and* $(\tilde{f}_1)$*–*$(\tilde{f}_4)$ *are satisfied. Then*

$$\begin{aligned}\mathcal{J}_b(v) &\geq \mathcal{J}_b(sv^++tv^-)+\frac{1-s^4}{4}\langle\mathcal{J}'_b(v),v^+\rangle+\frac{1-t^4}{4}\langle\mathcal{J}'_b(v),v^-\rangle\\ &\quad+\frac{(1-\theta_0)(1-s^2)^2}{4}\|v^+\|^2+\frac{(1-\theta_0)(1-t^2)^2}{4}\|v^-\|^2\\ &\quad+\frac{b(s^2-t^2)^2}{4}\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2,\quad\forall\, v=v^++v^-\in H,\quad s,t\geq 0.\end{aligned}\tag{2.1}$$

Proof. With $(\tilde{f}_4)$, for any $x\in\mathbb{R}^3,t\geq 0,\tau\in\mathbb{R}\backslash\{0\}$, we have

$$\begin{aligned}&K(x)\left[\frac{1-t^4}{4}\tau\tilde{f}(x,\tau)+\tilde{F}(x,t\tau)-\tilde{F}(x,\tau)\right]+\frac{\theta_0V(x)}{4}(1-t^2)^2\tau^2\\ &\quad=\int_t^1\left\{K(x)\left[\frac{\tilde{f}(x,\tau)}{\tau^3}-\frac{\tilde{f}(x,s\tau)}{(s\tau)^3}\right]+\theta_0V(x)\frac{(1-s^2)}{(s\tau)^2}\right\}s^3\tau^4\mathrm{d}s\geq 0.\end{aligned}\tag{2.2}$$

Hence, combining (1.17) and (1.18), for any $s, t \geq 0$, we imply that

$$\begin{aligned}
&\mathcal{J}_b(v) - \mathcal{J}_b(sv^+ + tv^-) \\
&\quad = \frac{1}{2}(\|v\|^2 - \|sv^+ + tv^-\|^2) + \frac{b}{4}\left(\|\nabla v\|_2^4 - \|s\nabla v^+ + t\nabla v^-\|_2^4\right) \\
&\qquad + \int_{\mathbb{R}^3} K(x)\left[\tilde{F}(x, sv^+ + tv^-) - \tilde{F}(x, v)\right] \mathrm{d}x \\
&\quad = \frac{1-s^4}{4}\left(\|v^+\|^2 + b\|\nabla v^+\|_2^4\right) + \frac{1-t^4}{4}\left(\|v^-\|^2 + b\|\nabla v^-\|_2^4\right) \\
&\qquad + \frac{(1-s^2)^2}{4}\|v^+\|^2 + \frac{(1-t^2)^2}{4}\|v^-\|^2 + \frac{b(1-s^2t^2)}{2}\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2 \\
&\qquad + \int_{\mathbb{R}^3} K(x)\left[\tilde{F}(x, sv^+) + \tilde{F}(x, tv^-) - \tilde{F}(x, v^+) - \tilde{F}(x, v^-)\right] \mathrm{d}x \\
&\quad = \frac{1-s^4}{4}\langle \mathcal{J}'_b(v), v^+\rangle + \frac{1-t^4}{4}\langle \mathcal{J}'_b(v), v^-\rangle + \frac{(1-s^2)^2}{4}\|v^+\|^2 \\
&\qquad + \frac{(1-t^2)^2}{4}\|v^-\|^2 + \frac{b(s^2-t^2)^2}{2}\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2 \\
&\qquad + \int_{\mathbb{R}^3} K(x)\left[\frac{1-s^4}{4}\tilde{f}(x, v^+)v^+ + \tilde{F}(x, sv^+) - \tilde{F}(x, v^+)\right] \mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^3} K(x)\left[\frac{1-t^4}{4}\tilde{f}(x, v^-)v^- + \tilde{F}(x, tv^-) - \tilde{F}(x, v^-)\right] \mathrm{d}x.
\end{aligned}$$

By (2.2) and the above inequality, we get

$$\begin{aligned}
&\mathcal{J}_b(v) - \mathcal{J}_b(sv^+ + tv^-) \\
&\quad \geq \frac{1-s^4}{4}\langle \mathcal{J}'_b(v), v^+\rangle + \frac{1-t^4}{4}\langle \mathcal{J}'_b(v), v^-\rangle + \frac{(1-\theta_0)(1-s^2)^2}{4}\|v^+\|^2 \\
&\qquad + \frac{(1-\theta_0)(1-t^2)^2}{4}\|v^-\|^2 + \frac{b(s^2-t^2)^2}{2}\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2 \\
&\qquad + \int_{\mathbb{R}^3}\left\{K(x)\left[\frac{1-s^4}{4}\tilde{f}(x, v^+)v^+ + \tilde{F}(x, sv^+) - \tilde{F}(x, v^+)\right]\right. \\
&\qquad\qquad \left. + \frac{\theta_0 V(x)}{4}(1-s^2)^2|v^+|^2\right\} \mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^3}\left\{K(x)\left[\frac{1-t^4}{4}\tilde{f}(x, v^-)v^- + \tilde{F}(x, tv^-) - \tilde{F}(x, v^-)\right]\right. \\
&\qquad\qquad \left. + \frac{\theta_0 V(x)}{4}(1-t^2)^2|v^-|^2\right\} \mathrm{d}x \\
&\quad \geq \frac{1-s^4}{4}\langle \mathcal{J}'_b(v), v^+\rangle + \frac{1-t^4}{4}\langle \mathcal{J}'_b(v), v^-\rangle + \frac{(1-\theta_0)(1-s^2)^2}{4}\|v^+\|^2 \\
&\qquad + \frac{(1-\theta_0)(1-t^2)^2}{4}\|v^-\|^2 + \frac{b(s^2-t^2)^2}{2}\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2,
\end{aligned}$$

which implies that (2.1) holds. □

Corollary 2.4. *Suppose that* $(V),(K)$, *and* $(\tilde{f}_1)$*–*$(\tilde{f}_4)$ *are satisfied. If* $v = v^+ + v^- \in \mathcal{M}_b$, *then*

$$\begin{aligned}\mathcal{J}_b(v) \ &\geq \mathcal{J}_b(sv^+ + tv^-) + \frac{(1-\theta_0)(1-s^2)^2}{4}\|v^+\|^2 + \frac{(1-\theta_0)(1-t^2)^2}{4}\|v^-\|^2 \\ &\quad + \frac{b(s^2-t^2)^2}{4}\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2, \quad \forall\, s,t \geq 0. \end{aligned} \tag{2.3}$$

Corollary 2.5. *Suppose that* $(V),(K)$, *and* $(\tilde{f}_1)$*–*$(\tilde{f}_4)$ *are satisfied. If* $v = v^+ + v^- \in \mathcal{M}_b$, *then*

$$\mathcal{J}_b(v^+ + v^-) = \max_{s,t\geq 0} \mathcal{J}_b(sv^+ + tv^-). \tag{2.4}$$

Lemma 2.6. *Suppose that* $(V),(K)$, *and* $(\tilde{f}_1)$*–*$(\tilde{f}_4)$ *are satisfied. If* $v \in H$ *with* $v^\pm \neq 0$, *then there exists a unique pair* (s_v, t_v) *of positive numbers, such that* $s_v v^+ + t_v v^- \in \mathcal{M}_b$.

Proof. We will first prove the existence of (s_v, t_v). Set

$$\begin{aligned} g_1(s,t) \ &= s^2\|v^+\|^2 + bs^4\|\nabla v^+\|_2^4 + bs^2t^2\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2 \\ &\quad - \int_{\mathbb{R}^3} K(x)\tilde{f}(x, sv^+)sv^+\mathrm{d}x \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} g_2(s,t) \ &= t^2\|v^-\|^2 + bt^4\|\nabla v^-\|_2^4 + bs^2t^2\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2 \\ &\quad - \int_{\mathbb{R}^3} K(x)\tilde{f}(x, tv^-)tv^-\mathrm{d}x. \end{aligned} \tag{2.6}$$

For any fixed $t \geq 0$, it follows from $(\tilde{f}_1)$ and $(\tilde{f}_3)$ that $g_1(s,s) > 0, g_1(s,s) > 0$ for $s > 0$ small enough and $g_1(t,t) < 0$ and $g_2(t,t) < 0$ for t large. Thus, there exist $0 < a_1 < a_2$, such that

$$g_1(a_1,a_1) > 0, \quad g_2(a_1,a_1) > 0, \quad g_1(a_2,a_2) < 0, \quad g_2(a_2,a_2) < 0. \tag{2.7}$$

From (2.5)–(2.7), we have

$$g_1(a_1,t) > 0, \quad g_1(a_2,t) < 0 \quad \forall\, t \in [a_1,a_2] \tag{2.8}$$

and

$$g_2(s,a_1) > 0, \quad g_2(s,a_2) < 0 \quad \forall\, s \in [a_1,a_2]. \tag{2.9}$$

By Miranda's Theorem [31], there exists a pair (s_u, t_u) with $a_1 < s_v, t_v < a_2$, such that $g_1(s_v,t_v) = g_2(s_v,t_v) = 0$. Hence, $s_v v^+ + t_v v^- \in \mathcal{M}_b$.

Next, we prove the uniqueness. Let (s_1,t_1) and (s_2,t_2) be such that $s_i v^+ + t_i v^- \in \mathcal{M}_b$, where $i = 1,2$. Taking the advantage of Corollary 2.4, we have

$$\begin{aligned} \mathcal{J}_b(s_1v^+ + t_1v^-) \geq \mathcal{J}_b(s_2v^+ + t_2v^-) + \frac{(1-\theta_0)(s_1^2-s_2^2)^2}{4s_1^2}\|v^+\|^2 \\ + \frac{(1-\theta_0)(t_1^2-t_2^2)^2}{4t_1^2}\|v^-\|^2 \end{aligned}$$

and

$$\mathcal{J}_b(s_2v^+ + t_2v^-) \geq \mathcal{J}_b(s_1v^+ + t_1v^-) + \frac{(1-\theta_0)(s_1^2 - s_2^2)^2}{4s_2^2}\|v^+\|^2 + \frac{(1-\theta_0)(t_1^2 - t_2^2)^2}{4t_2^2}\|v^-\|^2,$$

which implies that $(s_1, t_1) = (s_2, t_2)$. □

Lemma 2.7. *Suppose that* $(V), (K)$ *and* $(\tilde{f}_1)$*–*$(\tilde{f}_4)$ *are satisfied. Then*

$$\inf_{v\in\mathcal{M}_b} \mathcal{J}_b(v) = m_b = \inf_{v\in H, v^\pm\neq 0} \max_{s,t\geq 0} \mathcal{J}_b(sv^+ + tv^-).$$

Proof. By Corollary 2.5, we obtain

$$\begin{aligned}\inf_{u\in H, v^\pm\neq 0} \max_{s,t\geq 0} \mathcal{J}_b(sv^+ + tv^-) &\leq \inf_{v\in\mathcal{M}_b} \max_{s,t\geq 0} \mathcal{J}_b(sv^+ + tv^-)\\ &= \inf_{v\in\mathcal{M}_b} \mathcal{J}_b(v) = m_b \end{aligned}\tag{2.10}$$

Moreover, for any $v \in H$ with $v^\pm \neq 0$, it follows from Lemma 2.6 that:

$$\max_{s,t\geq 0} \mathcal{J}_b(sv^+ + tv^-) \geq \mathcal{J}_b(sv^+ + tv^-) \geq \inf_{v\in\mathcal{M}_b} \mathcal{J}_b(v) = m_b,$$

which implies

$$\inf_{v\in H, v^\pm\neq 0} \max_{s,t\geq 0} \mathcal{J}_b(sv^+ + tv^-) \geq \inf_{v\in\mathcal{M}_b} \mathcal{J}_b(v) = m_b. \tag{2.11}$$

Hence, combining (2.10) and (2.11), we have that the conclusion holds. □

Lemma 2.8. *Suppose that* $(\tilde{f}_4)$ *is satisfied. Then, for any* $\tau \in \mathbb{R}$,

$$K(x)\left[\frac{1}{4}\tau\tilde{f}(x,\tau) - \tilde{F}(x,\tau)\right] + \frac{\theta_0 V(x)}{4}\tau^2 \geq 0. \tag{2.12}$$

Proof. Taking $t = 0$ in (2.2), we can easily get the conclusion. □

Lemma 2.9. *Suppose that* $(V), (K)$*, and* $(\tilde{f}_1)$*–*$(\tilde{f}_4)$ *are satisfied. Then,* $m_b > 0$ *can be achieved.*

Proof. Let $\{v_n\} \subset \mathcal{M}_b$ be such that $\mathcal{J}_b(v_n) \to m_b$. According to (1.17), (1.18) and (2.12), for large $n \in \mathbb{N}$, one has

$$\begin{aligned}&1 + m_b\\ &\geq \mathcal{J}_b(v_n) - \frac{1}{4}\langle \mathcal{J}'_b(v_n), v_n\rangle\\ &\geq \frac{1-\theta_0}{4}\|v_n\|^2 + \int_{\mathbb{R}^3}\left\{K(x)\left[\frac{1}{4}\tilde{f}(x,v_n)v_n - \tilde{F}(x,v_n)\right] + \frac{\theta_0 V(x)}{4}|v_n|^2\right\}\mathrm{d}x\\ &\geq \frac{1-\theta_0}{4}\|v_n\|^2. \end{aligned}\tag{2.13}$$

It shows that $\{v_n\}$ is bounded in H due to $0 < \theta_0 < 1$, and then, there exists a $v_b \in H$, such that $v_n^\pm \rightharpoonup v_b^\pm$ in H. Since $\langle \mathcal{J}'_b(v), v\rangle = 0, \forall\, v \in \mathcal{M}_b$, then by

$(\tilde{f}_1)$–$(\tilde{f}_3)$ and Sobolev embedding theorem, for any $\epsilon > 0$, we have

$$\begin{aligned}\|v\|^2 &\leq \int_{\mathbb{R}^3} (|\nabla v|^2 + V(x)v^2)\mathrm{d}x + b\left(\int_{\mathbb{R}^3} |\nabla v|^2 \mathrm{d}x\right)^2 \\ &= \int_{\mathbb{R}^3} K(x)f(x,v)v\mathrm{d}x \\ &\leq \epsilon \int_{\mathbb{R}^3} K(x)|v|^2\mathrm{d}x + C_\epsilon \int_{\mathbb{R}^3} K(x)|v|^6\mathrm{d}x \\ &\leq \epsilon C_1\|v\|^2 + C_2\|v\|^6,\end{aligned}$$

where C_1 and C_2 are positive constants. We can choose $\epsilon = \frac{1}{2C_1}$, so there exists a constant $\alpha > 0$, such that $\|v\|^2 \geq \alpha$. Moreover, by (V), (K), $(\tilde{f}_1)$–$(\tilde{f}_3)$, (1.18), and ([47], A.1), one can conclude that

$$\begin{aligned}0 < \alpha \leq \liminf_{n\to\infty} & \left(\|v_n^\pm\|^2 + b\textstyle\int_{\mathbb{R}^3} |\nabla v_n|^2\mathrm{d}x \int_{\mathbb{R}^3} |\nabla v_n^\pm|^2\mathrm{d}x\right) \\ &= \liminf_{n\to\infty} \textstyle\int_{\mathbb{R}^3} K(x)\tilde{f}(x,v_n^\pm)v_n^\pm\mathrm{d}x \\ &= \textstyle\int_{\mathbb{R}^3} K(x)\tilde{f}(x,v_b^\pm)v_b^\pm\mathrm{d}x + o(1),\end{aligned} \tag{2.14}$$

which yields that $v_b^\pm \neq 0$. In fact, if $v_b^\pm = 0$, then we have $0 < \alpha \leq 0$, which is contradiction. Furthermore, by (2.14), the weak semicontinuity of norm and Fatou's Lemma, we get

$$\begin{aligned}&\|v_b^\pm\|^2 + b\int_{\mathbb{R}^3} |\nabla v_b|^2\mathrm{d}x \int_{\mathbb{R}^3} |\nabla v_b^\pm|^2\mathrm{d}x \\ &\quad\leq \liminf_{n\to\infty}\left[\|v_n^\pm\|^2 + b\int_{\mathbb{R}^3} |\nabla v_n|^2\mathrm{d}x \int_{\mathbb{R}^3} |\nabla v_n^\pm|^2\mathrm{d}x\right] \\ &\quad= \int_{\mathbb{R}^3} K(x)\tilde{f}(x,v_b^\pm)v_b^\pm\mathrm{d}x.\end{aligned} \tag{2.15}$$

This shows that

$$\langle \mathcal{J}'_b(v_b), v_b^\pm\rangle \leq 0. \tag{2.16}$$

Thus, by (1.17), (1.18), (2.1), (2.12), and (2.16), the weak semicontinuity of norm, Fatou's Lemma, and Lemma 2.5, we obtain

$$\begin{aligned}m_b &= \lim_{n\to\infty}\left[\mathcal{J}_b(v_n) - \frac{1}{4}\langle \mathcal{J}'_b(v_n), v_n\rangle\right] \\ &= \lim_{n\to\infty}\left\{\frac{1}{4}\|v_n\|^2 + \int_{\mathbb{R}^3} K(x)\left[\frac{1}{4}\tilde{f}(x,v_n)v_n - \tilde{F}(x,v_n)\right]\mathrm{d}x\right\} \\ &\geq \frac{1}{4}\liminf_{n\to\infty}\left[\int_{\mathbb{R}^3} |\nabla v_n|^2\mathrm{d}x + (1-\theta_0)\int_{\mathbb{R}^3} V(x)|v_n|^2\mathrm{d}x\right] \\ &\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^3}\left\{K(x)\left[\frac{1}{4}\tilde{f}(x,v_n)v_n - \tilde{F}(x,v_n)\right] + \frac{\theta_0}{4}V(x)|v_n|^2\right\}\mathrm{d}x \\ &\geq \frac{1}{4}\left[\int_{\mathbb{R}^3} |\nabla v_b|^2\mathrm{d}x + (1-\theta_0)\int_{\mathbb{R}^3} V(x)|v_b|^2\mathrm{d}x\right] \\ &\quad + \int_{\mathbb{R}^3}\left\{K(x)\left[\frac{1}{4}\tilde{f}(x,v_b)v_b - \tilde{F}(x,v_b)\right] + \frac{\theta_0}{4}V(x)|v_b|^2\right\}\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\|v_b\|^2 + \int_{\mathbb{R}^3} K(x)\left[\frac{1}{4}\tilde{f}(x,v_b)v_b - \tilde{F}(x,v_b)\right]\mathrm{d}x \\
&= \mathcal{J}_b(v_b) - \frac{1}{4}\langle \mathcal{J}'{}_b(v_b), v_b\rangle \\
&\geq \sup_{s,t\geq 0}\left[\mathcal{J}_b(sv_b^+ + tv_b^-) + \frac{1-s^4}{4}\langle \mathcal{J}'{}_b(v_b), v_b^+\rangle + \frac{1-t^4}{4}\langle \mathcal{J}'{}_b(v_b), v_b^-\rangle\right] \\
&\quad - \frac{1}{4}\langle \mathcal{J}'{}_b(v_b), v_b\rangle \\
&\geq \sup_{s,t\geq 0} \mathcal{J}_b(sv_b^+ + tv_b^-) \geq m_b,
\end{aligned}
$$

which implies that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^3} |\nabla v_n|^2\mathrm{d}x = \int_{\mathbb{R}^3} |\nabla v_b|^2\mathrm{d}x \tag{2.17}
$$

and

$$
\lim_{n\to\infty}\int_{\mathbb{R}^3} V(x)|v_n|^2\mathrm{d}x = \int_{\mathbb{R}^3} V(x)|v_b|^2\mathrm{d}x. \tag{2.18}
$$

Hence, $v_n \to v_b$ in H, then we know that $\mathcal{J}_b(v_b) = m_b$ and $v_b \in \mathcal{M}_b$. □

Lemma 2.10. *Suppose that $(V), (K)$, and $(\tilde{f}_1)$–$(\tilde{f}_4)$ are satisfied. If $v_0 \in \mathcal{M}_b$ and $\mathcal{J}_b(v_0) = m_b$, then v_0 is a critical point of $\mathcal{J}_b$.*

Proof. By contradiction, let $v_0 = v_0^+ + v_0^- \in \mathcal{M}_b, \mathcal{J}_b(v_0) = m_b$ and $\mathcal{J}'{}_b(v_0) \neq 0$. Then, there exist $\delta > 0$ and $\sigma > 0$, such that

$$
v \in H,\ \|v - v_0\| \leq 3\delta \Rightarrow \|\mathcal{J}_b(v)\| \geq \sigma. \tag{2.19}
$$

By Corollary 2.4, one has

$$
\begin{aligned}
\mathcal{J}_b(sv_0^+ + tv_0^-) &\leq \mathcal{J}_b(v_0) - \frac{(1-\theta_0)(1-s^2)^2}{4}\|v_0^+\|^2 - \frac{(1-\theta_0)(1-t^2)^2}{4}\|v_0^-\|^2 \\
&= m_b - \frac{(1-\theta_0)(1-s^2)^2}{4}\|v_0^+\|^2 - \frac{(1-\theta_0)(1-t^2)^2}{4}\|v_0^-\|^2.
\end{aligned} \tag{2.20}
$$

Let $D = (0.5, 1.5) \times (0.5, 1.5)$. It follows from (2.20) that:

$$
\kappa := \max_{(s,t)\in\partial D} \mathcal{J}_b(sv_0^+ + tv_0^-) < m_b. \tag{2.21}
$$

For $\varepsilon := \min\{(m_b - \kappa)/3, 1, \sigma\delta/8\}, S := B(u_0, \delta)$, ([47], Lemma 2.3) yields a deformation $\eta \in \mathcal{C}([0,1] \times H, H)$, such that

(i) $\eta(1, v) = v$ if $v \notin \mathcal{J}_b^{-1}([m_b - 2\varepsilon, m_b + 2\varepsilon]) \cap S_{2\delta}$.
(ii) $\eta(1, \mathcal{J}_b^{m_b+\varepsilon} \cap B(v_0, \delta)) \subset \mathcal{J}_b^{m_b-\varepsilon}$.
(iii) $\mathcal{J}_b(\eta(1, v)) \leq \mathcal{J}_b(v),\ \forall\, v \in H$.

By Corollary 2.5, $\mathcal{J}_b(sv_0^+ + tv_0^-) \leq \mathcal{J}_b(v_0) = m_b$ for $s, t \geq 0$, then it follows from iii) that:

$$
\mathcal{J}_b(\eta(1, sv_0^+ + tv_0^-)) \leq m_b - \varepsilon, \quad \forall\, s,t \geq 0, \quad |s-1|^2 + |t-1|^2 < \delta^2/\|v_0\|^2. \tag{2.22}
$$

On the other hand, by (iii) and (2.20), for any $s,t\geq 0$, $|s-1|^2+|t-1|^2\geq \delta^2/\|v_0\|^2$, one has

$$\begin{aligned}\mathcal{J}(\eta(1,sv_0^++tv_0^-)) &\leq \mathcal{J}_b(sv_0^++tv_0^-)\\ &\leq m_b-\frac{(1-\theta_0)(1-s^2)^2}{4}\|v_0^+\|^2-\frac{(1-\theta_0)(1-t^2)^2}{4}\|v_0^-\|^2\\ &\leq m_b-\frac{(1-\theta_0)\delta^2}{8\|v_0\|^2}\min\{\|v_0^+\|^2,\|v_0^-\|^2\}. \qquad (2.23)\end{aligned}$$

Combining (2.22) with (2.23), we get

$$\max_{(s,t)\in\bar{D}}\mathcal{J}_b(\eta(1,sv_0^++tv_0^-))<m_b. \qquad (2.24)$$

Moreover, $g(s,t):=sv_0^++tv_0^-$. By an argument similar as [4,41,42], we can get $\eta(1,g(D))\cap\mathcal{M}_b\neq\emptyset$. Since $m_b:=\inf_{v\in\mathcal{M}_b}\mathcal{J}_b(v)$, this is a contradiction. The proof is completed. □

3. Sign-changing solutions

Proof of Theorem 1.1. By (f_1)–(f_3), we know that $(\tilde{f}_1)$–$(\tilde{f}_3)$ hold. In view of Lemmas 2.9 and 2.10, there exists $v_b\in\mathcal{M}_b$, such that $\mathcal{J}_b(v_b)=m_b$ and $\mathcal{J}'_b(v_b)=0$. Thus, v_b is a sign-changing solution of (1.1). Next, we prove that v_b has exactly two nodal domains. Let $v_b=v_1+v_2+v_3$, where

$$v_1\geq 0,\quad v_2\leq 0,\quad \Omega_1\cap\Omega_2=\emptyset,\quad v_1|_{\Omega_2\cup\Omega_3}=v_2|_{\Omega_1\cap\Omega_3}=v_3|_{\Omega_1\cap\Omega_2}=0, \qquad (3.1)$$

$$\begin{aligned}&\Omega_1:=\{x\in\mathbb{R}^3: v_1(x)>0\},\quad \Omega_2:=\{x\in\mathbb{R}^3: v_2(x)<0\},\\ &\qquad\Omega_3:=\mathbb{R}^3\backslash(\Omega_1\cup\Omega_2),\end{aligned} \qquad (3.2)$$

where Ω_1 and Ω_2 are connected open subsets of $\mathbb{R}^3$.

Setting $w=v_1+v_2$, we see that $w^+=v_1$ and $w^-=v_2$, i.e., $w^\pm\neq 0$. By (1.17), (1.18), (2.1), (2.12), and (3.1), we have

$$\begin{aligned}m_b&=\mathcal{J}_b(v_b)=\mathcal{J}_b(v_b)-\frac{1}{4}\langle\mathcal{J}'_b(v_b),v_b\rangle\\ &=\mathcal{J}_b(w)+\mathcal{J}_b(v_3)+\frac{b}{2}\|\nabla w\|_2^2\|\nabla v_3\|_2^2\\ &\quad-\frac{1}{4}\left[\langle\mathcal{J}'_b(w),w\rangle+\langle\mathcal{J}'_b(w_3),w_3\rangle+2b\|\nabla w\|_2^2\|\nabla v_3\|_2^2\right]\\ &\geq\sup_{s,t\geq 0}\left[\mathcal{J}_b(sw^++tw^-)+\frac{1-s^4}{4}\langle\mathcal{J}'_b(w),w^+\rangle+\frac{1-t^4}{4}\langle\mathcal{J}'_b(w),w^-\rangle\right]\\ &\quad-\frac{1}{4}\langle\mathcal{J}'_b(w),w\rangle+\mathcal{J}_b(v_3)-\frac{1}{4}\langle\mathcal{J}'_b(v_3),v_3\rangle\\ &\geq\sup_{s,t\geq 0}\left[\mathcal{J}_b(sw^++tw^-)+\frac{bs^4}{4}\|\nabla w^+\|_2^2\|\nabla v_3\|_2^2+\frac{bt^4}{4}\|\nabla w^-\|_2^2\|\nabla v_3\|_2^2\right]\\ &\quad+\frac{1}{4}\|v_3\|^2+\int_{\mathbb{R}^3}K(x)\left[\frac{1}{4}\tilde{f}(x,v_3)v_3-\tilde{F}(x,v_3)\right]\mathrm{d}x\end{aligned}$$

$$\geq \sup_{s,t\geq 0} \mathcal{J}_b(sw^+ + tw^-) + \frac{(1-\theta_0)}{4}\|v_3\|^2$$
$$\geq m_b + \frac{(1-\theta_0)}{4}\|v_3\|^2,$$

which implies that $v_3 = 0$. Therefore, v_b has exactly two nodal domains. □

4. Ground state solutions of Nehari type

In this section, we will use non-Nehari manifold method to seek ground state solutions of Nehari type for (1.1). Before stating our results, we need to state the following lemmas and corollaries.

Lemma 4.1. *Suppose that* $(V),(K)$, *and* $(\tilde{f}_1)$*–*$(\tilde{f}_4)$ *are satisfied. Then*

$$\mathcal{J}_b(v) \geq \mathcal{J}_b(tv) + \frac{1-t^4}{4}\langle \mathcal{J}'_b(v), v\rangle + \frac{(1-\theta_0)(1-t^2)^2}{4}\|v\|^2, \quad \forall\, v \in H, \quad t \geq 0. \tag{4.1}$$

Corollary 4.2. *Suppose that* $(V),(K)$, *and* $(\tilde{f}_1)$*–*$(\tilde{f}_4)$ *are satisfied. Then, for any* $v \in \mathcal{N}_b$

$$\mathcal{J}_b(v) \geq \mathcal{J}_b(tv) + \frac{(1-\theta_0)(1-t^2)^2}{4}\|v\|^2, \quad \forall\, t \geq 0. \tag{4.2}$$

Corollary 4.3. *Suppose that* $(V),(K)$, *and* $(\tilde{f}_1)$*–*$(\tilde{f}_4)$ *are satisfied. Then, for any* $v \in \mathcal{N}_b$

$$\mathcal{J}_b(v) = \max_{t\geq 0} \mathcal{J}_b(tv). \tag{4.3}$$

Lemma 4.4. *Suppose that* $(V),(K)$, *and* $(\tilde{f}_1)$*–*$(\tilde{f}_4)$ *are satisfied. If* $v \in H\backslash\{0\}$, *then there exists a unique* $t_v > 0$, *such that* $t_v v \in \mathcal{N}_b$.

Lemma 4.5. *Suppose that* $(V),(K)$, *and* $(\tilde{f}_1)$*–*$(\tilde{f}_4)$ *are satisfied. Then*

$$\inf_{v\in\mathcal{N}_b} \mathcal{J}_b(v) = c_b = \inf_{v\in H, v\neq 0} \max_{t\geq 0} \mathcal{J}_b(tv).$$

Lemma 4.6. *Suppose that* $(V),(K)$, *and* $(\tilde{f}_1)$*–*$(\tilde{f}_4)$ *are satisfied. Then, there exist a constant* $c_* \in (0, c_b]$ *and a sequence* $\{v_n\} \subset H$ *satisfying*

$$\mathcal{J}_b(v_n) \to c_*, \quad \|\mathcal{J}'_b(v_n)\|(1 + \|v_n\|) \to 0. \tag{4.4}$$

Proof. Since $(\tilde{f}_1),(\tilde{f}_2)$, and (1.18) hold, then there exist $\delta_0 > 0$ and $\rho_0 > 0$, such that

$$v \in H, \quad \|v\| = \delta_0 \Rightarrow \mathcal{J}_b(v) \geq \rho_0. \tag{4.5}$$

Choose $w_k \in \mathcal{N}_b$, such that

$$c_b \leq \mathcal{J}_b(w_k) < c_b + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{4.6}$$

Since $\mathcal{J}_b(w_k) < 0$ for large $t > 0$, then according to [12] and Mountain Pass Lemma, we can derive that there exists a sequence $\{v_{k,n}\}_{n\in\mathbb{N}} \subset H$ satisfying

$$\mathcal{J}_b(v_{k,n}) \to c_k, \quad \|\mathcal{J}'_b(v_{k,n})\|(1 + \|v_{k,n}\|) \to 0, \quad k \in \mathbb{N}, \tag{4.7}$$

where $c_k \in [\rho_0, \sup_{t\geq 0} \mathcal{J}_b(tw_k)]$. By Corollary 4.5, one has

$$\mathcal{J}_b(w_k) \geq \mathcal{J}_b(tw_k), \quad \forall\, t \geq 0,$$

which implies that $\mathcal{J}_b(w_k) = \sup_{t\geq 0} \mathcal{J}_b(tw_k)$. Hence, by (4.6) and (4.9), we have

$$\mathcal{J}_b(v_{k,n}) < c_b + \frac{1}{k}, \quad \|\mathcal{J}'{}_b(v_{k,n})\|(1 + \|v_{k,n}\|) \to 0, \quad k \in \mathbb{N}. \tag{4.8}$$

Now, we can choose a sequence $\{n_k\} \subset \mathbb{N}$, such that

$$\mathcal{J}_b(v_{k,n_k}) < c_b + \frac{1}{k}, \quad \|\mathcal{J}'{}_b(v_{k,n_k})\|(1 + \|v_{k,n_k}\|) < \frac{1}{k}, \quad k \in \mathbb{N}. \tag{4.9}$$

Let $v_k = v_{k,n_k}, k \in \mathbb{N}$. Then, up to a subsequence, we have

$$\mathcal{J}_b(v_n) \to c_* \in [\rho_0, c_b], \quad \|\mathcal{J}'{}_b(v_n)\|(1 + \|v_n\|) \to 0.$$

This completes the proof. □

Proof of Theorem 1.2. By (f_1)–(f_3), we know that $(\tilde{f}_1)$–$(\tilde{f}_3)$ hold. From Lemma 4.8, we can deduce that there exists a sequence $\{v_n\} \subset H$ satisfying (4.4), such that

$$\mathcal{J}_b(v_n) \to c_*, \quad \langle \mathcal{J}'{}_b(v_n), v_n \rangle \to 0. \tag{4.10}$$

From (1.17), (1.18), (2.12), and (4.10), one has for large $n \in \mathbb{N}$

$$c_* + 1 \geq \mathcal{J}_b(v_n) - \frac{1}{4}\langle \mathcal{J}'{}_b(v_n), v_n \rangle \geq \frac{1-\theta_0}{4}\|v_n\|^2.$$

This implies that $\{v_n\}$ is bounded in H. By a standard argument, we can prove that there exists a $v_0 \in H\backslash\{0\}$, such that $\mathcal{J}'{}_b(v_0) = 0$. This shows that $v_0 \in \mathcal{N}_b$ is a non-trivial solution of (1.1) and $\mathcal{J}_b(v_0) \geq c_b$. On the other hand, using (1.17), (1.18), (2.12), the weak semicontinuity of norm, and Fatou's Lemma, we have

$$\begin{aligned}
c_b \geq c_* &= \lim_{n\to\infty} \left(\mathcal{J}_b(v_n) - \frac{1}{4}\langle \mathcal{J}'{}_b(v_n), v_n \rangle \right) \\
&= \lim_{n\to\infty} \left[\frac{1}{4}\|v_n\|^2 + \int_{\mathbb{R}^3} K(x) \left(\frac{1}{4}\tilde{f}(x, v_n)v_n - \tilde{F}(x, v_n) \right) \right] \\
&\geq \frac{1}{4} \liminf_{n\to\infty} \left(\int_{\mathbb{R}^3} |\nabla v_n|^2 \mathrm{d}x + (1-\theta_0) \int_{\mathbb{R}^3} V(x)|v_n|^2 \mathrm{d}x \right) \\
&\quad + \liminf_{n\to\infty} \int_{\mathbb{R}^3} \left\{ K(x) \left(\frac{1}{4}\tilde{f}(x, v_n)v_n - \tilde{F}(x, v_n) \right) + \frac{\theta_0 V(x)}{4}|v_n|^2 \right\} \mathrm{d}x \\
&\geq \frac{1}{4} \left(\int_{\mathbb{R}^3} |\nabla v_0|^2 \mathrm{d}x + (1-\theta_0) \int_{\mathbb{R}^3} V(x)|v_0|^2 \mathrm{d}x \right) \\
&\quad + \int_{\mathbb{R}^3} \left\{ K(x) \left(\frac{1}{4}\tilde{f}(x, v_0)v_0 - \tilde{F}(x, v_0) \right) + \frac{\theta_0 V(x)}{4}|v_0|^2 \right\} \mathrm{d}x \\
&= \frac{1}{4}\|v_0\|^2 + \int_{\mathbb{R}^3} K(x) \left(\frac{1}{4}\tilde{f}(x, v_0)u_0 - \tilde{F}(x, v_0) \right) \mathrm{d}x \\
&= \mathcal{J}_b(v_0) - \frac{1}{4}\langle \mathcal{J}'{}_b(v_0), v_0 \rangle = \mathcal{J}_b(v_0).
\end{aligned}$$

Hence, we have $\mathcal{J}_b(v_0) \leq c_*$, and so, $\mathcal{J}_b(v_0) = c_b = \inf_{\mathcal{N}_b} \mathcal{J}_b > 0$.

From Theorem 1.1, there exists a $v_b \in \mathcal{M}_b$, such that $\mathcal{J}_b(v_b) = m_b$. Thus, by (1.16), Corollary 2.5, and Lemma 4.7, one has

$$\begin{aligned} m_b = \mathcal{J}_b(v_b) &= \sup_{s,t\geq 0} \mathcal{J}_b(sv_b^+ + tv_b^-) \\ &= \sup_{s,t\geq 0}\left[\mathcal{J}_b(sv_b^+) + \mathcal{J}_b(tv_b^-) + \frac{bs^2t^2}{2}\|\nabla v_b^+\|_2^2\|\nabla v_b^-\|_2^2\right] \\ &> \sup_{s\geq 0}\mathcal{J}_b(sv_b^+) + \sup_{t\geq 0}\mathcal{J}_b(tv_b^-) \\ &\geq 2c_b. \end{aligned}$$

The proof is completed. □

5. The convergence property

Now, we are in a situation to give the proof of Theorem 1.3.

Proof of Theorem 1.3. By (f_1)–(f_3), we know that $(\tilde{f}_1)$–$(\tilde{f}_3)$ hold. In Sect. 2, $b = 0$ is allowed in the argument. Therefore, under the assumptions of Theorem 1.3, there exists a $w_0 \in \mathcal{M}_0$, such that $\mathcal{J}'_0(w_0) = 0$ and $\mathcal{J}_0(w_0) = m_0 = \inf_{v\in\mathcal{M}_0}\mathcal{J}_0(v)$, that is (1.4), has a least energy sign-changing solution, which changes sign only once.

For any $b > 0$, let $v_b \in \mathcal{M}_b$ be a sign-changing solution of (1.1) obtained in Theorem 1.2, which changes sign only once and satisfies $\mathcal{J}_b(v_b) = m_b$.

Choose $\vartheta_0 \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, such that $\vartheta_0^\pm \neq 0$. From (K) and $(\tilde{f}_1)$–$(\tilde{f}_3)$, there exist $\beta_1 > 0$ and $\beta_2 \geq \max\{\|\nabla\vartheta_0^+\|_2^4, \|\nabla\vartheta_0^-\|_2^4\}$, such that for any $s, t \in \mathbb{R}$

$$\begin{aligned} &\int_{\mathbb{R}^3} K(x)F(x, s\vartheta_0^+)\mathrm{d}x \geq \beta_2|s|^4 - \beta_1, \\ &\int_{\mathbb{R}^3} K(x)F(x, t\vartheta_0^-)\mathrm{d}x \geq \beta_2|t|^4 - \beta_1. \end{aligned} \tag{5.1}$$

For any $b \in [0, 1]$, it follows from (1.17) and Lemma 2.5 that:

$$\begin{aligned} \mathcal{J}_b(v_b) &= m_b \leq \max_{s,t\geq 0}\mathcal{J}_b(s\vartheta_0^+ + t\vartheta_0^-) \\ &= \max_{s,t\geq 0}\left\{\frac{s^2}{2}\|\vartheta_0^+\|^2 + \frac{bs^4}{4}\|\nabla\vartheta_0^+\|_2^4 - \int_{\mathbb{R}^3} K(x)F(x, s\vartheta_0^+)\mathrm{d}x\right. \\ &\quad + \frac{t^2}{2}\|\vartheta_0^-\|^2 + \frac{bt^4}{4}\|\nabla\vartheta_0^-\|_2^4 - \int_{\mathbb{R}^3} K(x)F(x, t\vartheta_0^-)\mathrm{d}x \\ &\quad \left.+ \frac{bs^2t^2}{2}\|\nabla\vartheta_0^+\|_2^2\|\nabla\vartheta_0^-\|_2^2\right\} \\ &\leq \max_{s,t\geq 0}\left\{\frac{s^2}{2}\|\omega_0^+\|^2 + \frac{bs^4}{2}\|\nabla\vartheta_0^+\|_2^4 + 2\beta_1 - \beta_2 s^4 + \frac{t^2}{2}\|\vartheta_0^-\|^2\right. \\ &\quad \left.+ \frac{bt^4}{2}\|\nabla\vartheta_0^-\|_2^4 - \beta_2 t^4 + \frac{bs^2t^2}{2}\|\nabla\vartheta_0^-\|_2^2\|\nabla\vartheta_0^-\|_2^2\right\} \\ &\leq \max_{s\geq 0}\left[\frac{s^2}{2}\|\vartheta_0^+\|^2 - \frac{s^4}{2}\|\nabla\vartheta_0^+\|_2^4\right] \end{aligned}$$

$$
\begin{aligned}
&+\max_{t\geq 0}\left[\frac{t^2}{2}\|\vartheta_0^-\|^2-\frac{t^4}{2}\|\nabla\vartheta_0^-\|_2^4\right]+2\beta_1\\
&:=\Lambda_0>0.
\end{aligned}
$$

By (1.17), (1.18), and (2.12), we get

$$
\Lambda_0+1\geq \mathcal{J}_{b_n}(v_{b_n})-\frac{1}{4}\langle \mathcal{J}'_{b_n}(v_{b_n}),v_{b_n}\rangle\geq\frac{(1-\theta_0)}{4}\|v_{b_n}\|^2,
$$

which implies that $\{v_{b_n}\}$ is bounded in H. Then, there exists a subsequence of $\{b_n\}$, still denoted by $\{b_n\}$, and $v_0\in H$, such that $v_{b_n}\rightharpoonup v_0$ in H. Similar to Lemma 2.9, we conclude that $v_{b_n}^{\pm}\to v_0^{\pm}\neq 0$ in H. Note that

$$
\begin{aligned}
\langle \mathcal{J}'_0(v_0),\varphi\rangle&=\int_{\mathbb{R}^3}(\nabla v_0\nabla\varphi+V(x)v_0\varphi)\mathrm{d}x-\int_{\mathbb{R}^3}K(x)\tilde{f}(x,v_0)\varphi\mathrm{d}x\\
&=\lim_{n\to\infty}\left[\int_{\mathbb{R}^3}(\nabla v_{b_n}\nabla\varphi+V(x)v_{b_n}\varphi)\mathrm{d}x-\int_{\mathbb{R}^3}K(x)\tilde{f}(x,v_{b_n})\varphi\mathrm{d}x\right]\\
&=\lim_{n\to\infty}\langle \mathcal{J}'_{b_n}(v_{b_n}),\varphi\rangle=0\quad\forall\,\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^3).
\end{aligned}
$$

This shows that $\mathcal{J}'_0(v_0)=0$, and then, $v_0\in\mathcal{M}_0$ and $\mathcal{J}_0(v_0)\geq m_0$. Next, we claim that $\mathcal{J}_0(v_0)=m_0$. Let $b_n\in[0,1]$, then it follows from (K) and $(\tilde{f}_3)$ that there exists $K_0>0$, such that for all $s\geq K_0$ or $t\geq K_0$,

$$
\begin{aligned}
&\mathcal{J}_{b_n}(sw_0^++tw_0^-)\\
&\quad=\frac{s^2}{2}\|w_0^+\|^2+\frac{b_ns^4}{4}\|\nabla w_0^+\|_2^4-\int_{\mathbb{R}^3}K(x)\tilde{F}(x,sw_0^+)\mathrm{d}x+\frac{t^2}{2}\|w_0^-\|^2\\
&\qquad+\frac{b_nt^4}{4}\|\nabla w_0^-\|_2^4-\int_{\mathbb{R}^3}K(x)\tilde{F}(x,tw_0^-)\mathrm{d}x+\frac{b_ns^2t^2}{2}\|\nabla w_0^+\|_2^2\|\nabla w_0^-\|_2^2\\
&\quad\leq\frac{s^2}{2}\|w_0^+\|^2+\frac{s^4}{2}\|\nabla w_0^+\|_2^4-\int_{\mathbb{R}^3}K(x)\tilde{F}(x,sw_0^+)\mathrm{d}x\\
&\qquad+\frac{t^2}{2}\|w_0^-\|^2+\frac{t^4}{2}\|\nabla w_0^-\|_2^4-\int_{\mathbb{R}^3}K(x)\tilde{F}(x,tw_0^-)\mathrm{d}x\\
&\quad<0.
\end{aligned}\tag{5.2}
$$

By Lemma 2.6, there exists (s_n,t_n), such that $s_nw_0^++t_nw_0^-\in\mathcal{M}_{b_n}$ which, together with (5.2), implies $0<s_n,t_n<K_0$. Hence, from (1.17), (1.18), and (2.1), we have

$$
\begin{aligned}
m_0&=\mathcal{J}_0(w_0)\\
&=\mathcal{J}_{b_n}(w_0)-\frac{b_n}{4}\|\nabla w_0\|_2^4\\
&\geq\mathcal{J}_{b_n}(s_nw_0^++t_nw_0^-)+\frac{1-s_n^4}{4}\langle\mathcal{J}'_{b_n}(w_0),w_0^+\rangle+\frac{1-t_n^4}{4}\langle\mathcal{J}'_{b_n}(w_0),w_0^-\rangle\\
&\quad-\frac{b_n}{4}\|\nabla w_0\|_2^4\\
&\geq m_{b_n}-\frac{1+K_0^4}{4}\left|\langle\mathcal{J}'_{b_n}(w_0),w_0^+\rangle\right|-\frac{1+K_0^4}{4}\left|\langle\mathcal{J}'_{b_n}(w_0),w_0^-\rangle\right|\\
&\quad-\frac{b_n}{4}\|\nabla w_0\|_2^4
\end{aligned}
$$

$$= m_{b_n} - \frac{(1+K_0^4)b_n}{4}\|\nabla w_0\|_2^2\|\nabla w_0^+\|_2^2 - \frac{(1+K_0^4)b_n}{4}\|\nabla w_0\|_2^2\|\nabla w_0^-\|_2^2 - \frac{b_n}{4}\|\nabla w_0\|_2^4,$$

which implies that

$$\limsup_{n\to\infty} m_{b_n} \le m_0. \tag{5.3}$$

By (1.17) and (5.3), one has

$$m_0 \le \mathcal{J}_0(v_0) = \limsup_{n\to\infty} \mathcal{J}_{b_n}(v_{b_n}) = \limsup_{n\to\infty} m_{b_n} \le m_0.$$

This shows that $\mathcal{J}_0(v_0) = m_0$. □

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