Bialgebra structures on flat Lie algebras

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Abstract

In this paper, we investigate Lie bialgebra structures on nondegenerate flat Lie algebras, which are solvable Lie algebras admitting an orthogonal decomposition into an abelian subalgebra and an abelian ideal. These algebras correspond precisely to the Lie algebras of Lie groups endowed with a flat left-invariant Riemannian metric. We establish a decomposition theorem for 1-cocycles, showing that any 1-cocycle can be expressed as the sum of a coboundary and a cocycle mapping the abelian subalgebra into the space of invariant bivectors. Using the Big Bracket formalism ; an algebraic framework that efficiently encodes Lie bialgebra structures, we obtain a classification of such structures on nondegenerate flat Lie algebras. Finally, we illustrate these structures through a series of examples.

Keywords: Lie bialgebras, Poisson-Lie groups, flat Lie algebras, Yang-Baxter equation, Big Bracket.

1. Introduction

Lie bialgebras, first introduced by Drinfeld in 1983 [5], are fundamental algebraic structures that arise in the study of Poisson-Lie groups and quantum groups. They constitute the infinitesimal counterpart of Poisson-Lie group structures and play an important role in the quantization of these structures.

A Poisson-Lie group is a Lie group endowed with a Poisson structure, such that its multiplication is a Poisson map.The Lie algebra of a Poisson-Lie group is called a *Lie bialgebra*.

A Lie bialgebra consists of a Lie algebra $\mathfrak{g}$ along with a 1-cocycle $\xi : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ such that the transpose map $\xi^t : \bigwedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$ defines a Lie bracket on the dual space $\mathfrak{g}^*$. Lie bialgebras play an important role in the theory of integrable systems and noncommutative geometry.

An important class of solvable Lie algebras are the flat Lie algebras, which arise naturally in the study of homogeneous Riemannian manifolds with flat left invariant metrics. Milnor characterized flat Lie algebras as being 2-step solvable with an orthogonal decomposition into an abelian subalgebra and an abelian ideal [10]. Further refinements in [3],[2] led to the decomposition as $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ where $\mathfrak{s}$ is an abelian subalgebra, $\mathfrak{z}$ is the center, and its commutator ideal $[\mathfrak{g}, \mathfrak{g}]$ is abelian even-dimensional.

The study of Lie bialgebra structures on flat Lie algebras is motivated by several factors:

- Connections to deformation quantization: In the context of deformation quantization, Poisson-Lie groups play a crucial role in establishing connections between classical and noncommutative geometry. The flatness condition investigated in this work has been

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shown by Hawkins [7],[8] to be related to deformations issues arising in formal deformation quantization of Poisson structures and therefore one should expect that quantizing Poisson-Lie groups which integrates the Lie bialgebras listed here should produce quantum groups with a good behaviours in terms of noncommutative geometry, following the framework developed by Connes.

- Expanding classification results: The classification of Lie bialgebra structures on specific classes of Lie algebras is an important problem that provides insights into the corresponding Poisson-Lie group structures. For semisimple Lie algebras, Belavin and Drinfeld provided a complete classification in terms of r-matrices satisfying the classical Yang-Baxter equation [4]. Other classifications have been obtained for low dimensional Lie algebras [11],[15]. However, for general solvable Lie algebras, the classification problem becomes much more challenging due to the lack of powerful representation-theoretic tools available in the semisimple case.

In this paper, we use the Big Bracket formalism to facilitate our study of Lie bialgebras on flat Lie algebras. The Big Bracket is an algebraic tool used to encode the structure of a Lie bialgebra in a compact form, which simplifies calculations. Using this formalism, we aim to provide a solution to the open problem of classifying all possible Lie bialgebra structures that can arise on nondegenerate flat Lie algebras. This work is motivated by our prior research [1].

Main result.. We establish a decomposition theorem for cocycles on flat Lie algebras, which states that any cocycle can be expressed as the sum of a coboundary and a specific type of cocycle. The latter cocycle maps elements from its abelian subalgebra to its set of invariant bivectors. Furthermore, we provide explicit descriptions of Lie bialgebra structures for various classes of flat Lie algebras using the Big Bracket formalism.

The paper is organized as follows: In Section 2, we review the necessary background on Lie bialgebras and flat Lie algebras, including their characterization and normal forms. Section 3 presents our main results, including the theorem on the decomposition of cocycles and the classification of Lie bialgebra structures. In Section 4, we discuss several examples illustrating the application of our results to specific classes of flat Lie algebras.

The results developed in this work may provide a starting point for investigating deformations of these structures, their cohomological properties, and potential relationships with other classes of Lie bialgebras.

2. Preliminaries

2.1. Lie Bialgebras

Let $(\mathfrak{g}, \mu)$ be a real Lie algebra and let $\xi : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ be a 1-cocycle with respect to the adjoint representation, meaning that for all $x, y \in \mathfrak{g}$

$$\xi([x,y]) = \mathrm{ad}_x\ \xi(y) - \mathrm{ad}_y\ \xi(x).$$

Here, the adjoint action is extended to $\bigwedge^2 \mathfrak{g}$ by setting

$$\mathrm{ad}_x(y \wedge z) = \mathrm{ad}_x(y) \wedge z + y \wedge \mathrm{ad}_x(z)$$

for every $x, y, z \in \mathfrak{g}$.

The pair $(\mathfrak{g}, \xi)$ is called a *Lie bialgebra* if the transpose

$$\xi^t : \bigwedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$$

defines a Lie bracket on the dual space $\mathfrak{g}^*$. This structure serves as the infinitesimal counterpart to the Poisson-Lie group structure (see [12]).

A linear map $\varphi : (\mathfrak{g}, \xi) \to (\mathfrak{h}, \delta)$ is called a *morphism of Lie bialgebras* if it is a Lie algebra homomorphism and satisfies

$$\delta \circ \varphi = (\varphi \wedge \varphi) \circ \xi.$$

This condition is succinctly expressed by the commutative diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi"] \arrow[d, "\xi"'] & \mathfrak{h} \arrow[d, "\delta"] \\ \bigwedge^2 \mathfrak{g} \arrow[r, "\varphi\wedge\varphi"'] & \bigwedge^2 \mathfrak{h}. \end{tikzcd}$$

2.1.1. Characteristic Derivation

For a Lie bialgebra $(\mathfrak{g}, \mu, \xi)$, the *characteristic derivation* is an endomorphism of $\mathfrak{g}$ defined as the composition

$$D_\xi \colon \mathfrak{g} \xrightarrow{\xi} \bigwedge^2 \mathfrak{g} \xrightarrow{\mu} \mathfrak{g}.$$

This map captures how the cobracket interact with the Lie structure. It is a derivation of the Lie algebra $\mathfrak{g}$

$$D_\xi([x, y]) = [D_\xi(x), y] + [x, D_\xi(y)] \text{ for all } x, y \in \mathfrak{g},$$

and its transpose acts as a derivation on the dual Lie algebra $\mathfrak{g}^*$. Moreover, D_ξ is invariant under Lie bialgebra isomorphisms [6].

Example. Let $\mathfrak{g} = \operatorname{span}\{e_1, e_2, e_3\}$ be the Lie algebra with nontrivial brackets

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2.$$

Consider the family of Lie bialgebra structures on $\mathfrak{g}$ defined by the 1-cocycles

$$\begin{aligned} \xi(e_1) &= a\, e_1 \wedge e_2 + b\, e_1 \wedge e_3, \\ \xi(e_2) &= c\, e_1 \wedge e_2 + b\, e_2 \wedge e_3, \\ \xi(e_3) &= c\, e_1 \wedge e_3 - a\, e_2 \wedge e_3. \end{aligned}$$

The corresponding characteristic derivation is represented by the matrix

$$D = \begin{pmatrix} 0 & 0 & 0 \\ -b & 0 & -c \\ a & c & 0 \end{pmatrix}$$

Its characteristic polynomial is $p(\lambda) = -\lambda^3 - c^2\lambda$. So distinct values of c yield non-isomorphic bialgebras.

2.2. Flat Lie algebras

A Lie algebra $\mathfrak{g}$ (real of finite dimension) is said to be *flat* if it is endowed with a positive definite scalar product $\langle\,,\rangle$ for which the associated infinitesimal Levi-Civita connection, defined for all x, y, z in $\mathfrak{g}$ by

$$2\langle\nabla_x y, z\rangle = \langle[x,y],z\rangle + \langle[z,x],y\rangle + \langle[z,y],x\rangle \tag{1}$$

has zero curvature

$$R(x,y,z) = \nabla_{[x,y]}z - (\nabla_x\nabla_y z - \nabla_y\nabla_x z) \equiv 0. \tag{2}$$

In other words, the associated Lie group G of $\mathfrak{g}$, endowed with the unique left invariant Riemannian metric extending $\langle\,,\rangle$ has its Levi-Civita connection flat.

Proposition 2.1 ([2], [3], [10]). *Let $(\mathfrak{g}, \langle\,,\rangle)$ be a flat Lie algebra. Then $\mathfrak{g}$ decomposes orthogonally as*

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z} \oplus [\mathfrak{g},\mathfrak{g}],$$

where $\mathfrak{z}$ is the center of $\mathfrak{g}$, $\mathfrak{s}$ is an abelian Lie subalgebra, $[\mathfrak{g},\mathfrak{g}]$ is the commutator ideal satisfying the following conditions:

- $[\mathfrak{g},\mathfrak{g}]$ *is abelian and even dimensional,*
- $\mathrm{ad}_x = \nabla_x$, *for any x in $\mathfrak{z} \oplus \mathfrak{s}$.*

Moreover, from [2], we have the following *normal form* of a flat Lie algebra:

$$\mathfrak{g} = \mathrm{span}\{s_1,\dots,s_{k_0}\} \oplus \mathrm{span}\{z_1,\dots,z_{\ell_0}\} \oplus \mathrm{span}\{d_1,\dots,d_{2m}\} \tag{3}$$

where $\mathrm{span}\{d_1,\dots,d_{2m}\}$ is the commutator ideal of $\mathfrak{g}$ which is abelian, $\mathrm{span}\{z_1,\dots,z_{\ell_0}\}$ its center (possibly trivial) and $\mathrm{span}\{s_1,\dots,s_{k_0}\}$ its abelian subalgebra such that $k_0 \le m$ and

$$[s_i, d_{2j-1}] = \lambda_{ij}\, d_{2j},\ \ [s_i, d_{2j}] = -\lambda_{ij}\, d_{2j-1} \text{ for all } i = 1,\dots,k_0, \text{ and } j = 1,\dots,m. \tag{4}$$

Examples 2.2.

1. Any commutative Lie algebra is flat.
2. The Poincaré algebra corresponding to 2-dimensional Minkowski space, is a 3-dimensional Lie algebra with basis $\{s, d_1, d_2\}$, where s generates the translations and d_1, d_2 generate the Lorentz transformations. The Lie brackets are given by:

   $$[s,d_1] = d_2,\ \ [s,d_2] = -d_1,\ \ [d_1,d_2] = 0,$$

   This algebra constitutes the lowest-dimensional non-abelian flat Lie algebra and serves as the Lie algebra of the Poincaré group, which is the isometry group of 2-dimensional Minkowski spacetime.

Henceforth, we shall focus on noncommutative flat Lie algebras.

Remarks 2.3. 1. A flat Lie algebra $\mathfrak{g}$ is unimodular, 2-step solvable whose nilradical is precisely $\mathfrak{z} \oplus [\mathfrak{g},\mathfrak{g}]$.

2. A Lie algebra is called *complete* if its centre is trivial and its derivations are all inner. However, a flat Lie algebra is not complete as there is a derivation that maps the subalgebra $\mathfrak{s}$ to zero while acting as the identity on the derived subalgebra $[\mathfrak{g},\mathfrak{g}]$, and this derivation is not inner.

2.2.1. Isomorphism Class

Two flat Lie algebras $\big(\mathfrak{g}_1, \langle \cdot, \cdot \rangle_1\big)$ and $\big(\mathfrak{g}_2, \langle \cdot, \cdot \rangle_2\big)$, are said to be *isomorphic* if there exists a linear map $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ that is both a Lie algebra isomorphism and an isometry with respect to the given inner products.

Let φ be an automorphism of a flat Lie algebra $\big(\mathfrak{g}, \langle \cdot, \cdot \rangle\big)$, endowed with a fixed scalar product and a normal basis as specified in (4). Since φ is an automorphism, one has

$$\varphi\big(\mathfrak{z}\big) = \mathfrak{z} \quad \text{and} \quad \varphi\big([\mathfrak{g}, \mathfrak{g}]\big) = [\mathfrak{g}, \mathfrak{g}],$$

which implies that $\varphi(\mathfrak{s}) = \mathfrak{s}$. Consequently, one may express

$$\varphi(s_i) = \sum_{p=1}^{k_0} \alpha_{pi}\, s_p,$$

where the matrix (α_{pi}) is invertible. Moreover, we have

$$\mathrm{ad}_{\varphi(s_i)}(d_{2j-1}) = \Lambda_{ij}\, d_{2j} \quad \text{and} \quad \mathrm{ad}_{\varphi(s_i)}(d_{2j}) = -\Lambda_{ij}\, d_{2j-1},$$

with

$$\Lambda_{ij} = \sum_{p=1}^{k_0} \lambda_{pj}\, \alpha_{pi}.$$

It follows that the restriction

$$\mathrm{ad}_{\varphi(s_i)}\big|_{[\mathfrak{g},\mathfrak{g}]}^{2}$$

is represented by a diagonal matrix whose eigenvalues are $\{-\Lambda_{i1}^2, \ldots, -\Lambda_{im}^2\}$; each eigenvalue $-\Lambda_{ij}^2$ has multiplicity two, with corresponding eigenvectors d_{2j-1} and d_{2j}. On the other hand, one also obtains

$$\begin{cases} \mathrm{ad}^2_{\varphi(s_i)}\, \varphi(d_{2j-1}) = -\lambda_{ij}^2\, \varphi(d_{2j-1}), \\ \mathrm{ad}^2_{\varphi(s_i)}\, \varphi(d_{2j}) = -\lambda_{ij}^2\, \varphi(d_{2j}). \end{cases}$$

Thus, there exists a permutation $\sigma \in S_m$ such that for every $j = 1, \ldots, m$, the automorphism φ maps the plane

$$P_j = \mathrm{Span}\{d_{2j-1}, d_{2j}\}$$

onto $P_{\sigma(j)}$, and the restriction $\varphi\big|_{P_j}$ is given by a 2×2 orthogonal matrix.

The *characteristic matrix* of a flat Lie algebra is defined in terms of its structural constants (see (4)). After adopting the convention that the rows correspond to the vectors

$$L_j = (\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{k_0 j}), \quad j = 1, \ldots, m,$$

the characteristic matrix is given by

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{21} & \cdots & \lambda_{k_0 1} \\ \lambda_{12} & \lambda_{22} & \cdots & \lambda_{k_0 2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1m} & \lambda_{2m} & \cdots & \lambda_{k_0 m} \end{pmatrix}. \tag{5}$$

A key property of this matrix is that it defines an injective linear map; in particular, no row is identically zero. This property is a direct consequence of the relation

$$\mathfrak{z} \cap [\mathfrak{g}, \mathfrak{g}] = \{0\}.$$

A change of orthonormal basis in $[\mathfrak{g}, \mathfrak{g}]$ that preserves the block-diagonal structure of the maps ad_{s_k} is effected precisely by the following transformations:

1. Permutations of the planes
$$P_i = \mathrm{Span}\{d_{2i-1}, d_{2i}\}, \quad i = 1, \dots, m;$$
2. Rotations of each plane P_i by an orthogonal matrix
$$O_i \in \mathrm{O}(2).$$

Under these operations, the characteristic matrix Λ is transformed by permuting its row vectors (i.e., replacing L_i by $L_{\sigma(i)}$, for some $\sigma \in S_m$) and by replacing a row L_i with $-L_i$ whenever the corresponding orthogonal transformation satisfies $\det O_i = -1$.

Moreover, one may perform a change of basis in $[\mathfrak{g}, \mathfrak{g}]$ by suitably permuting the planes P_i so that the first k_0 row vectors of Λ, denoted $L_1, \dots, L_{k_0}$, become linearly independent. Simultaneously, a corresponding change of basis in $\mathfrak{s}$ is applied so that these vectors are transformed into the canonical basis. As a consequence, the characteristic matrix Λ assumes the form

$$\Lambda = \begin{pmatrix} \mathrm{Id}_{k_0} \\ \Lambda_1 \end{pmatrix}.$$

From this point onward, we fix this ordered orthonormal basis of $\mathfrak{g}$, and all subsequent computations are carried out relative to it.

2.2.2. Nondegenerate Flat Lie Algebras

Let $\mathfrak{g}$ be a flat Lie algebra with a normal basis as in (4). The Lie algebra $\mathfrak{g}$ is said to be *degenerate* if there exist indices $i, j, k \in \{1, \dots, m\}$ such that

$$L_k = a\, L_i + b\, L_j, \quad \text{with } a, b \in \{-1, 1\}, \tag{6}$$

where L_i denotes the i-th row vector of the characteristic matrix Λ. If no such relation exists, $\mathfrak{g}$ is referred to as *nondegenerate*.

Remarks 2.4.

1. The flat Lie algebra $\mathfrak{g}_\alpha = \mathrm{span}\{s\} \oplus \mathrm{span}\{d_1, d_2, d_3, d_4\}$ with the brackets
$$[s, d_1] = d_2,\ [s, d_2] = -d_1,\ [s, d_3] = \alpha d_4,\ [s, d_4] = -\alpha d_3,\ \alpha \neq 0$$
is nondegenerate for $\alpha \neq -1, 1$ and degenerate for $\alpha = \pm 1$.
2. The definition above is coordinate-free. Indeed, the condition (6) remains invariant under any automorphism of flat Lie algebras, as demonstrated earlier (see [1]).
3. The nondegeneracy condition ensures that certain matrices arising in our analysis of cocycles will be invertible, which significantly simplifies the classification problem.

3. Main Theorems

3.1. Notations

Let $\mathfrak{g}$ be a flat Lie algebra with its decomposition given by (3):

$$\mathfrak{g} = \mathrm{span}\{s_1, \dots, s_{k_0}\} \oplus \mathrm{span}\{z_1, \dots, z_{\ell_0}\} \oplus \mathrm{span}\{d_1, \dots, d_{2m}\}.$$

Let $\xi : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ be a cocyle on $\mathfrak{g}$. The coefficients of $\xi(s_k)$, $\xi(z_k)$, $\xi(d_{2k-1})$, $\xi(d_{2k})$ are denoted respectively $a_{ij}^k, a_{ij}^{((k))}, a_{ij}^{(2k-1)}, a_{ij}^{(2k)} \dots$ and so on, similarly for a bivector $r \in \bigwedge^2 \mathfrak{g}$, namely

$$\begin{aligned}\xi(s_k) = &\sum_{1\le i<j\le k_0} a_{ij}^k\, s_i \wedge s_j + \sum_{\substack{1\le i\le k_0\\1\le j\le \ell_0}} b_{ij}^k\, s_i \wedge z_j + \sum_{\substack{1\le i\le k_0\\1\le j\le m}} c_{ij}^k\, s_i \wedge d_{2j-1} + \sum_{\substack{1\le i\le k_0\\1\le j\le m}} e_{ij}^k\, s_i \wedge d_{2j}\\ &+ \sum_{1\le i<j\le \ell_0} f_{ij}^k\, z_i \wedge z_j + \sum_{\substack{1\le i\le \ell_0\\1\le j\le m}} g_{ij}^k\, z_i \wedge d_{2j-1} + \sum_{\substack{1\le i\le \ell_0\\1\le j\le m}} h_{ij}^k\, z_i \wedge d_{2j}\\ &+ \sum_{1\le i<j\le m} m_{ij}^k\, d_{2i-1} \wedge d_{2j-1} + \sum_{1\le i,j\le m} n_{ij}^k\, d_{2i-1} \wedge d_{2j} + \sum_{1\le i<j\le m} p_{ij}^k\, d_{2i} \wedge d_{2j}.\end{aligned}$$

For all $k = 1, \dots, m$, $\mathrm{ad}_{d_{2k-1}} r = \Phi_k \wedge d_{2k}$ and $\mathrm{ad}_{d_{2k}} r = d_{2k-1} \wedge \Phi_k$, where

$$\begin{aligned}\Phi_k = &\left(\sum_{j=2}^{k_0} -\lambda_{jk} a_{1j}\right) s_1 + \sum_{p=2}^{k_0-1} \left(\sum_{i=1}^{p-1} \lambda_{ik} a_{ip} + \sum_{j=p+1}^{k_0} -\lambda_{jk} a_{pj}\right) s_p + \left(\sum_{i=1}^{k_0-1} \lambda_{ik} a_{ik_0}\right) s_{k_0}\\ &+ \sum_{j=1}^{\ell_0} \left(\sum_{i=1}^{k_0} \lambda_{ik} b_{ij}\right) z_j + \sum_{j=1}^{m} \left(\sum_{i=1}^{k_0} \lambda_{ik} c_{ij}\right) d_{2j-1} + \sum_{j=1}^{m} \left(\sum_{i=1}^{k_0} \lambda_{ik} e_{ij}\right) d_{2j}.\end{aligned}$$

3.2. Cocycles of a flat Lie algebra

Consider a flat nondegenerate Lie algebra $\mathfrak{g}$ with its normal basis as specified in (4). Let Λ represent its characteristic matrix given in (5). Denote the rows of Λ as L_i for $i = 1, \dots, m$, and let P_i be the planes generating the commutator ideal $[\mathfrak{g}, \mathfrak{g}] = \oplus_{i=1}^m P_i$. The following theorem outlines all possible 1-cocycles on $\mathfrak{g}$.

Theorem 3.1. *Let $\xi : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ be a cocycle on a flat nondegenerate Lie algebra $\mathfrak{g}$. Then ξ can be uniquely decomposed as*

$$\xi = \mathrm{ad}\, r + R, \tag{7}$$

where $r \in \bigwedge^2 \mathfrak{g}$ and $R : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ is a cocycle such that

- $R(\mathfrak{s}) \subset \left(\bigwedge^2 \mathfrak{g}\right)^{\mathfrak{g}}$
- $R(P_\ell) \subset (\mathfrak{s} \wedge P_\ell) \oplus (\mathfrak{z} \wedge P_\ell)$ *for all* $\ell = 1, \dots, m$

Proof.
- We first establish that any cocycle restricted to $\mathfrak{s}$ can be written as $\mathrm{ad}\, r_0 + R$, wher R takes values in invariant bivectors.
- We then extend R to a cocycle on the entire $\mathfrak{g}$ by analyzing the cocycle condition on the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$.

- Finally, we show that for nondegenerate flat Lie algebras, the components of the cocycle R in $\bigwedge^2[\mathfrak{g},\mathfrak{g}]$ vanish identically, which completes the characterization.

Let $\{s_1^*,\dots,s_{k_0}^*,z_1^*,\dots,z_{\ell_0}^*,d_1^*,\dots,d_{2m}^*\}$ be the dual basis of the dual vector space $\mathfrak{g}^*$. We apply successively 2-forms $\alpha\in\bigwedge^2\mathfrak{g}^*$ to the cocycle identities, i.e.

$$\begin{aligned}\alpha\left(\lambda_{k\ell}\xi(d_{2\ell})\right)&=\alpha\left(\operatorname{ad}_{s_k}\xi(d_{2\ell-1})\right)-\alpha\left(\operatorname{ad}_{d_{2\ell-1}}\xi(s_k)\right)\\-\alpha\left(\lambda_{k\ell}\xi(d_{2\ell-1})\right)&=\alpha\left(\operatorname{ad}_{s_k}\xi(d_{2\ell})\right)-\alpha\left(\operatorname{ad}_{d_{2\ell}}\xi(s_k)\right).\end{aligned}$$

We apply successively $s_i^*\wedge d_{2\ell}^*$, $s_i^*\wedge d_{2\ell-1}^*$, $z_i^*\wedge d_{2\ell}^*$, $z_i^*\wedge d_{2\ell}^*$ and we get

$$\left\{\begin{array}{rcl}\lambda_{k\ell}e_{i\ell}^{(2\ell)}&=&\lambda_{k\ell}c_{i\ell}^{(2\ell-1)}-s_i^*\left(\Phi_\ell\right),\\-\lambda_{k\ell}c_{i\ell}^{(2\ell-1)}&=&-\lambda_{k\ell}e_{i\ell}^{(2\ell)}+s_i^*\left(\Phi_\ell\right)\end{array}\right.,\quad\left\{\begin{array}{rcl}\lambda_{k\ell}h_{i\ell}^{(2\ell)}&=&\lambda_{k\ell}g_{i\ell}^{(2\ell-1)}-z_i^*\left(\Phi_\ell\right),\\-\lambda_{k\ell}g_{i\ell}^{(2\ell-1)}&=&-\lambda_{k\ell}h_{i\ell}^{(2\ell)}+z_i^*\left(\Phi_\ell\right)\end{array}\right.$$

i.e. $e_{i\ell}^{(2\ell)}=c_{i\ell}^{(2\ell-1)}$, $h_{i\ell}^{(2\ell)}=g_{i\ell}^{(2\ell-1)}$, $s_i^*\left(\Phi_\ell\right)=0$ and $z_i^*\left(\Phi_\ell\right)=0$. But we have

$$\Big(s_i^*\left(\Phi_\ell\right)\Big)_{\substack{1\le\ell\le m\\1\le i\le k_0}}=\begin{pmatrix}\lambda_{11}&\lambda_{21}&\dots&\lambda_{k_01}\\\lambda_{12}&\lambda_{22}&\dots&\lambda_{k_02}\\&\dots\dots\dots\dots&&\\\lambda_{1m}&\lambda_{2m}&\dots&\lambda_{k_0m}\end{pmatrix}\begin{pmatrix}0&a_{12}^k&\cdots&a_{1k_0}^k\\-a_{12}^k&0&\cdots&a_{2k_0}^k\\\vdots&\vdots&\ddots&\vdots\\-a_{1k_0}^k&-a_{2k_0}^k&\cdots&0\end{pmatrix}.$$

We deduce from the injectivity of Λ that $a_{ij}^k\equiv0$ and similarly $b_{ij}^k\equiv0$.

We have for all $k,\ell=1,\dots,k_0$, $\operatorname{ad}_{s_k}\xi(s_\ell)=\operatorname{ad}_{s_\ell}\xi(s_k)$, namely

$$\lambda_{kj}c_{ij}^\ell-\lambda_{\ell j}c_{ij}^k=0.$$

This implies that the vectors $(c_{ij}^1,\dots,c_{ij}^{k_0})$ and $L_j=(\lambda_{1j},\dots,\lambda_{k_0j})$ are linearly dependent. So there exists a real number c_{ij} such that $c_{ij}^k=c_{ij}\lambda_{kj}$. Similarly, for the other coefficients: $e_{ij}^k=e_{ij}\lambda_{kj}$, $g_{ij}^k=g_{ij}\lambda_{kj}$, $h_{ij}^k=h_{ij}\lambda_{kj}$.

We have

$$\left\{\begin{array}{rcl}-\lambda_{\ell i}p_{ij}^k+\lambda_{\ell j}m_{ij}^k&=&-\lambda_{ki}p_{ij}^\ell+\lambda_{kj}m_{ij}^\ell,\\\lambda_{\ell j}p_{ij}^k-\lambda_{\ell i}m_{ij}^k&=&\lambda_{kj}p_{ij}^\ell-\lambda_{ki}m_{ij}^\ell\end{array}\right.,\quad\left\{\begin{array}{rcl}-\lambda_{\ell i}n_{ji}^k+\lambda_{\ell j}n_{ij}^k&=&-\lambda_{ki}n_{ji}^\ell+\lambda_{kj}n_{ij}^\ell,\\\lambda_{\ell j}n_{ji}^k-\lambda_{\ell i}n_{ij}^k&=&\lambda_{kj}n_{ji}^\ell-\lambda_{ki}n_{ij}^\ell\end{array}\right.$$

then

$$\left\{\begin{array}{rcl}\left(\lambda_{\ell j}-\lambda_{\ell i}\right)\left(p_{ij}^k+m_{ij}^k\right)&=&\left(\lambda_{kj}-\lambda_{ki}\right)\left(p_{ij}^\ell+m_{ij}^\ell\right),\\\left(\lambda_{\ell j}+\lambda_{\ell i}\right)\left(p_{ij}^k-m_{ij}^k\right)&=&\left(\lambda_{kj}+\lambda_{ki}\right)\left(p_{ij}^\ell-m_{ij}^\ell\right)\end{array}\right.$$

therefore

$$\left\{\begin{array}{rcl}p_{ij}^k&=&A_{ij}\lambda_{kj}+B_{ij}\lambda_{ki},\\m_{ij}^k&=&-B_{ij}\lambda_{kj}-A_{ij}\lambda_{ki}\end{array}\right.,\ \text{likewise}\left\{\begin{array}{rcl}n_{ji}^k&=&a_{ij}\lambda_{kj}+b_{ij}\lambda_{ki},\\n_{ij}^k&=&-b_{ij}\lambda_{kj}-a_{ij}\lambda_{ki}\end{array}\right.$$

So $\xi(x)=\operatorname{ad}_x r_0+R(x)$ for all $x\in\mathfrak{s}$ with

$$R(s_k)=\sum_{1\le i<j\le\ell_0}f_{ij}^k\,z_i\wedge z_j+\sum_{i=1}^m n_{ii}^k\,d_{2i-1}\wedge d_{2i},\ \text{and}$$

$$
\begin{aligned}
r_0 = \sum_{\substack{1\le i\le k_0\\1\le j\le m}} e_{ij}\, s_i\wedge d_{2j-1} + \sum_{\substack{1\le i\le k_0\\1\le j\le m}} -c_{ij}\, s_i\wedge d_{2j} + \sum_{\substack{1\le i\le \ell_0\\1\le j\le m}} h_{ij}\, z_i\wedge d_{2j-1} + \sum_{\substack{1\le i\le \ell_0\\1\le j\le m}} -g_{ij}\, z_i\wedge d_{2j}\\
+ \sum_{1\le i<j\le m} a_{ij}\, d_{2i-1}\wedge d_{2j-1} + \sum_{1\le i<j\le m} B_{ij}\, d_{2i-1}\wedge d_{2j} + \sum_{1\le i<j\le m} -A_{ij}\, d_{2j-1}\wedge d_{2i}\\
+ \sum_{1\le i<j\le m} -b_{ij}\, d_{2i}\wedge d_{2j}.
\end{aligned}
$$

Note that r_0 is uniquely determined mod $\mathfrak{s}$-invariant elements of $\bigwedge^2\mathfrak{g}$, namely

$$
r_0 \in (\mathfrak{s}\wedge[\mathfrak{g},\mathfrak{g}])\oplus(\mathfrak{z}\wedge[\mathfrak{g},\mathfrak{g}])\oplus\left(\bigoplus_{1\le i<j\le m} P_i\wedge P_j\right).
$$

We have

$$
\xi\big|_{\mathfrak{s}\oplus\mathfrak{z}} = \operatorname{ad} r_0 + R, \quad \text{with} \quad R\big(\mathfrak{s}\oplus\mathfrak{z}\big)\subset\big(\bigwedge^2\mathfrak{g}\big)^{\mathfrak{g}}.
$$

Define

$$
\widehat{R} = \xi - \operatorname{ad} r_0,
$$

which is a 1-cycle extending R to $[\mathfrak{g},\mathfrak{g}]$, uniquely determined modulo $\big(\bigwedge^2\mathfrak{g}\big)^{\mathfrak{s}}$. As a cocycle, we have

$$
\widehat{R}\big([\mathfrak{g},\mathfrak{g}]\big)\subset\mathfrak{g}\wedge[\mathfrak{g},\mathfrak{g}],
$$

and for every $k=1,\dots,k_0$ and $\ell=1,\dots,m$,

$$
\begin{cases}
\lambda_{k\ell}\,\widehat{R}(d_{2\ell}) = \operatorname{ad}_{s_k}\widehat{R}(d_{2\ell-1}),\\
-\lambda_{k\ell}\,\widehat{R}(d_{2\ell-1}) = \operatorname{ad}_{s_k}\widehat{R}(d_{2\ell}).
\end{cases}
\tag{8}
$$

Moreover, for all $1\le k,\ell\le m$, the following relations hold:

$$
\begin{cases}
\operatorname{ad}_{d_{2k-1}}\widehat{R}(d_{2\ell}) = \operatorname{ad}_{d_{2\ell}}\widehat{R}(d_{2k-1}),\\
\operatorname{ad}_{d_{2k-1}}\widehat{R}(d_{2\ell-1}) = \operatorname{ad}_{d_{2\ell-1}}R(d_{2k-1}),\\
\operatorname{ad}_{d_{2k}}\widehat{R}(d_{2\ell}) = \operatorname{ad}_{d_{2\ell}}\widehat{R}(d_{2k}).
\end{cases}
\tag{9}
$$

In particular, for any distinct indices $k,\ell\in\{1,\dots,m\}$, we obtain

$$
\sum_{i=1}^{k_0}\lambda_{ik}\,c_{i\ell}^{(2\ell-1)} = 0, \quad \sum_{i=1}^{k_0}\lambda_{ik}\,e_{i\ell}^{(2\ell-1)} = -\sum_{i=1}^{k_0}\lambda_{i\ell}\,e_{ik}^{(2k-1)}.
$$

In matrix form, if we denote by Λ the characteristic matrix and set

$$
C = \big(c_{i\ell}^{(2\ell-1)}\big)_{\substack{1\le i\le k_0\\1\le\ell\le m}} = \big(C_0\mid C_1\big),
$$

then ΛC is a diagonal matrix:

$$
\Lambda C = \begin{pmatrix}\mathrm{Id}_{k_0}\\ \hline \Lambda_1\end{pmatrix}\big(C_0\mid C_1\big) = \begin{pmatrix} * & 0 & \cdots & 0\\ 0 & * & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & *\end{pmatrix}.
$$

This implies that

$$C_0 = \begin{pmatrix} c_{11}^{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_{k_0k_0}^{(2k_0-1)} \end{pmatrix}, \quad C_1 = 0, \quad \text{and} \quad \Lambda_1 C_0 = 0;$$

that is,

$$c_{ij}^{(2j-1)} = 0 \quad \text{for all } i, j = 1, \ldots, k_0 \ (j \neq i) \quad \text{and for all } j = k_0+1, \ldots, m.$$

Similarly, if we define

$$E = \big(e_{i\ell}^{(2\ell-1)}\big)_{\substack{1\le i\le k_0 \\ 1\le \ell\le m}} = \big(E_0 \mid E_1\big),$$

then ΛE is skew-symmetric:

$$\Lambda E = \left(\frac{\mathrm{Id}_{k_0}}{\Lambda_1}\right)\big(E_0 \mid E_1\big) = \begin{pmatrix} S & -A^t \\ A & S' \end{pmatrix},$$

which shows that $E_0 = S$ and $E_1 = S\,\Lambda_1^t$. On the other hand, one has

$$\sum_{i=1}^{k_0} e_{i\ell}^{(2\ell-1)}\, s_i \wedge d_{2\ell} = \mathrm{ad}_{d_{2\ell-1}}\left(\sum_{i,j} a_{ij}\, s_i \wedge s_j\right)$$

if and only if

$$\big(E_0 \mid E_1\big) = S\big(\mathrm{Id}_{k_0} \mid \Lambda_1^t\big), \quad \text{with} \quad S = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1k_0} \\ -a_{12} & 0 & \cdots & a_{2k_0} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1k_0} & -a_{2k_0} & \cdots & 0 \end{pmatrix}.$$

Thus, we obtain $E_0 = S$ and $E_1 = S\,\Lambda_1^t$. In a similar fashion, the identity

$$\sum_{j=1}^{\ell_0} h_{j\ell}^{(2\ell-1)}\, z_j \wedge d_{2\ell} = \mathrm{ad}_{d_{2\ell-1}}\left(\sum b_{ij}\, s_i \wedge z_j\right)$$

holds if and only if

$$B = H_0 \quad \text{and} \quad H_1 = H_0 \Lambda_1^t,$$

where $B = \big(b_{ij}\big)_{1\le i,j\le \ell_0}$.

In summary, there exists a bivector

$$r_1 \in \big(\bigwedge^2 \mathfrak{s}\big) \oplus \big(\mathfrak{s} \wedge \mathfrak{z}\big)$$

such that

$$\widehat{R} = \mathrm{ad}\, r_1 + \overline{R},$$

where $\overline{R}$ is the non-exact cocycle extending R to $[\mathfrak{g},\mathfrak{g}]$. Henceforth, we will simply denote $\overline{R}$ by R.

For the coefficients of $R(d_{2\ell-1})$, $R(d_{2\ell})$ in $\bigwedge^2[\mathfrak{g},\mathfrak{g}]$ we derive the following systems from the cocycle condition:

$$\begin{pmatrix} \lambda_{k\ell} & 0 & \lambda_{kj} & -\lambda_{ki} \\ 0 & \lambda_{k\ell} & -\lambda_{ki} & \lambda_{kj} \\ \lambda_{kj} & -\lambda_{ki} & \lambda_{k\ell} & 0 \\ -\lambda_{ki} & \lambda_{kj} & 0 & \lambda_{k\ell} \end{pmatrix} \begin{pmatrix} m_{ij}^{(2\ell)} \\ p_{ij}^{(2\ell)} \\ n_{ij}^{(2\ell-1)} \\ n_{ji}^{(2\ell-1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_{k\ell} & 0 & -\lambda_{kj} & \lambda_{ki} \\ 0 & \lambda_{k\ell} & \lambda_{ki} & -\lambda_{kj} \\ -\lambda_{kj} & \lambda_{ki} & \lambda_{k\ell} & 0 \\ \lambda_{ki} & -\lambda_{kj} & 0 & \lambda_{k\ell} \end{pmatrix} \begin{pmatrix} m_{ij}^{(2\ell-1)} \\ p_{ij}^{(2\ell-1)} \\ n_{ij}^{(2\ell)} \\ n_{ji}^{(2\ell)} \end{pmatrix}.$$

The matrices $M(a,b,c) = \left(\begin{smallmatrix} a & 0 & b & c \\ 0 & a & c & b \\ b & c & a & 0 \\ c & b & 0 & a \end{smallmatrix}\right)$, where $a,b,c \in \mathbb{R}$, are commuting, diagonalizable and their eigenvalues are $a-b-c, a-b+c, a+b-c, a+b+c$ with corresponding eigenvectors

$$\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Since $\mathfrak{g}$ is nondegenerate, then 0 is not an eigenvalue of $M(\lambda_{k\ell}, \lambda_{kj}, -\lambda_{ki})$, for some k. Then all the coefficients of $R(d_{2\ell-1})$ and $R(d_{2\ell})$ in $P_i \wedge P_j$ are zero. $\square$

Remarks 3.2.
1. The decomposition $\xi = \operatorname{ad} r + R$ is unique.
2. This decomposition remains valid in the case of degenerate flat Lie algebras; however, the cocycle R may have non-zero coefficients in $\bigwedge^2[\mathfrak{g},\mathfrak{g}]$.

Having established the decomposition of cocycles on flat Lie algebras, we now introduce a powerful formalism that will allow us to characterize when such a cocycle defines a Lie bialgebra structure.

3.3. Cobrackets on flat Lie algebras

In this section, we will introduce the concept of the *Big Bracket* and utilize it to determine all possible cobrackets on nodegenerate flat Lie algebras.

3.3.1. Big brackets

The Big Bracket provides an elegant and powerful framework for studying Lie bialgebras by unifying the Lie algebra and Lie coalgebra structures into a single mathematical object. This approach allows us to express the compatibility conditions between these structures in a compact form, simplifying both theoretical analysis and concrete calculations.

Let V be a finite-dimensional vector space over the real field $\mathbb{K} = \mathbb{R}$, and let V^* be its dual vector space. We consider the exterior algebra

$$\bigwedge(V^* \oplus V) = \underset{k \geq -2}{\oplus} \left(\underset{p+q=k}{\oplus} \bigwedge^{q+1}(V^*) \otimes \bigwedge^{p+1}(V) \right),$$

We say that an element σ of $\bigwedge(V^* \oplus V)$ is of bidegree (p,q) and of degree $|\sigma| = p+q$ if it belongs to $\bigwedge^{(p,q)}(V) = \bigwedge^{q+1}(V^*) \otimes \bigwedge^{p+1}(V)$.

The *big bracket* is the graded Lie-algebra structure $[\![\cdot,\cdot]\!]$ on the graded vector space $\bigwedge(V^*\oplus V) = \oplus_{k\geq -2}\bigwedge^{(p,q)}(V)$

$$[\![\cdot,\cdot]\!] : \bigwedge^{(p,q)}(V) \times \bigwedge^{(p',q')}(V) \to \bigwedge^{(p+p',q+q')}(V)$$

uniquely determined by the following properties:

(i) $[\![\sigma,\sigma']\!] = 0$, if σ and σ' both belong to $\mathbb{K}\oplus V$ or to $\mathbb{K}\oplus V^*$.

(ii) for all $\sigma \in V$, $\sigma' \in V^*$, $[\![\sigma,\sigma']\!] = \sigma'(\sigma)$,

(iii) for all $\sigma \in \bigwedge(V\oplus V^*)$ of degree k, $[\![\sigma,\cdot]\!]$ is a graded derivation, i.e., for all σ' and σ'', of degree k' and k''

$$[\![\sigma,\sigma'\wedge\sigma'']\!] = [\![\sigma,\sigma']\!]\wedge\sigma'' + (-1)^{kk'}\sigma'\wedge[\![\sigma,\sigma'']\!].$$

Additionally, as a graded Lie algebra, we have the graded anticommutativity property: for all $\sigma,\sigma \in \wedge^k(V\oplus V^*)$ of degree k an k'

$$[\![\sigma,\sigma']\!] = -(-1)^{kk'}[\![\sigma',\sigma]\!], \tag{10}$$

and the graded Jacobi identity, i.e. $[\![\sigma,\cdot]\!]$ is a graded derivation of degree k,

$$[\![\sigma,[\![\sigma',\sigma'']\!]]\!] = [\![[\![\sigma,\sigma']\!],\sigma'']\!] + (-1)^{kk'}[\![\sigma',[\![\sigma,\sigma'']\!]]\!]. \tag{11}$$

The explicit formula for the big bracket of decomposable elements is as follows. For $\xi\otimes x \in \bigwedge^{q+1}(V^*)\otimes\bigwedge^{p+1}(V)$ and $\eta\otimes y \in \bigwedge^{m+1}(V^*)\otimes\bigwedge^{\ell+1}(V)$

$$\begin{aligned}[\![\xi\otimes x,\eta\otimes y]\!] = (-1)^{pm+p}&\sum_{j=0}^{p}(-1)^j\xi\wedge i_{x_j}\eta\otimes x_0\wedge\ldots\wedge\widehat{x_j}\wedge\ldots\wedge x_p\wedge y\\ &- (-1)^{pm+q}\sum_{j=0}^{\ell}(-1)^j i_{y_j}\xi\wedge\eta\otimes x\wedge y_0\wedge\ldots\wedge\widehat{y_j}\wedge\ldots\wedge y_\ell.\end{aligned}$$

Where $i_{x_j}\eta$ denotes the interior product of a form η with a vector x_j, and the hat sign denotes an omitted factor. For a detailed treatment and for useful formulas see [13] and [14].

This bracket satisfies graded anticommutativity and the graded Jacobi identity, making $\bigwedge(V^*\oplus V)$ a graded Lie algebra.

The central insight for Lie bialgebra theory is that the triplet (V,μ,ξ), where $\mu : \bigwedge^2 V \to V$ and $\xi : V \to \bigwedge^2 V$ are linear maps (viewed as elements in $\bigwedge^2 V^*\otimes V$ and $V^*\otimes\bigwedge^2 V$ respectively), defines a Lie bialgebra if and only if:

$$[\![\mu+\xi,\mu+\xi]\!] = 0. \tag{12}$$

This elegant equation encapsulates all the axioms of a Lie bialgebra:

- The Jacobi identity: $[\![\mu,\mu]\!] = 0$.
- The co-Jacobi identity: $[\![\xi,\xi]\!] = 0$.

- The cocycle condition: $[\![\mu, \xi]\!] = 0$.

We will use the following useful formula: For $x^*, y^* \in V^*$ and $r_1, r_2 \in \bigwedge^2 V$

$$[\![x^* \otimes r_1, y^* \otimes r_2]\!] = -x^* \otimes (r_2 \wedge i_{y^*} r_1) - y^* \otimes (r_1 \wedge i_{x^*} r_2). \tag{13}$$

Lemma 3.3. *Let $\mathfrak{g}$ be a flat Lie algebra with its normal basis* (4). *Let $\xi : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ be a 1-cocycle. Then the following are equivalent:*

1. $[\![\xi, \xi]\!] = 0$.
2. *For all $x \in \mathfrak{g}$, $F(x) = 0$ where*

$$F(x) = \sum_{p=1}^{k_0} \xi(s_p) \wedge i_{s_p^*} \xi(x) + \sum_{p=1}^{\ell_0} \xi(z_p) \wedge i_{z_p^*} \xi(x) + \sum_{p=1}^{m} \xi(d_{2p-1}) \wedge i_{d_{2p-1}^*} \xi(x) + \sum_{p=1}^{m} \xi(d_{2p}) \wedge i_{d_{2p}^*} \xi(x).$$

Proof. Let $\xi \in \mathfrak{g}^* \otimes \bigwedge^2 \mathfrak{g}$ be a 1-cocycle on a flat Lie algebra and decompose it as $\xi = \xi_1 + \ldots + \xi_4$, where ξ_i corresponds to the components of ξ in the normal basis. We have for any $\omega \in \bigwedge^3 \mathfrak{g}^*$

$$\begin{aligned}
[\![\xi, \xi]\!] (s_i \otimes \omega) &= \left([\![\xi_1, \xi_1]\!] + 2 \sum_{1 < j \leq 4} [\![\xi_1, \xi_j]\!] \right) (s_i \otimes \omega) \\
&= -2\omega \Big(\sum_{p=1}^{k_0} \xi(s_p) \wedge i_{s_p^*} \xi(s_i) + \sum_{p=1}^{\ell_0} \xi(z_p) \wedge i_{z_p^*} \xi(s_i) \\
&\quad + \sum_{p=1}^{m} \xi(d_{2p-1}) \wedge i_{d_{2p-1}^*} \xi(s_i) + \sum_{p=1}^{m} \xi(d_{2p}) \wedge i_{d_{2p}^*} \xi(s_i) \Big).
\end{aligned}$$

By considering $[\![\xi, \xi]\!](x \otimes \omega)$ for all x in the basis of $\mathfrak{g}$, we obtain the desired result. $\square$

We now explicitly formulate the quadratic equations arising from the co-Jacobi condition . In the tables below, coefficients associated with the cocycle R are indicated by superscripts.

ω	$\langle \omega, F(z_i) \rangle \ (i = 1, \ldots, \ell_0)$
$s_j^* \wedge d_{2k-1}^* \wedge d_{2k}^*$	$2c_{jk}^{(2k-1)} n_{kk}^{((i))}$
$z_j^* \wedge d_{2k-1}^* \wedge d_{2k}^*$	$2g_{jk}^{(2k-1)} n_{kk}^{((i))} + \sum_{1 \leq p < j} n_{kk}^{((p))} f_{pj}^{((i))} - \sum_{j < p \leq \ell_0} n_{kk}^{((p))} f_{jp}^{((i))}$
$z_j^* \wedge z_k^* \wedge z_\ell^*$	$\sum_{1 \leq p < \ell} f_{jk}^{((p))} f_{p\ell}^{((i))} - \sum_{\ell < p \leq \ell_0} f_{jk}^{((p))} f_{\ell p}^{((i))} - \sum_{1 \leq p < k} f_{j\ell}^{((p))} f_{pk}^{((i))}$ $+ \sum_{k < p \leq \ell_0} f_{j\ell}^{((p))} f_{kp}^{((i))} + \sum_{1 \leq p < j} f_{k\ell}^{((p))} f_{pj}^{((i))} - \sum_{j < p \leq \ell_0} f_{k\ell}^{((p))} f_{jp}^{((i))}$

ω	$\langle \omega, F(s_i) \rangle \ (i = 1, \ldots, k_0)$
$s_j^* \wedge d_{2k-1}^* \wedge d_{2k}^*$	$2c_{jk}^{(2k-1)} n_{kk}^{i}$
$z_j^* \wedge d_{2k-1}^* \wedge d_{2k}^*$	$2g_{jk}^{(2k-1)} n_{kk}^{i} + \sum_{1 \leq p < j} n_{kk}^{((p))} f_{pj}^{i} - \sum_{j < p \leq \ell_0} n_{kk}^{((p))} f_{jp}^{i}$
$z_j^* \wedge z_k^* \wedge z_\ell^*$	$\sum_{1 \leq p < \ell} f_{jk}^{((p))} f_{p\ell}^{i} - \sum_{\ell < p \leq \ell_0} f_{jk}^{((p))} f_{\ell p}^{i} - \sum_{1 \leq p < k} f_{j\ell}^{((p))} f_{pk}^{i}$ $+ \sum_{k < p \leq \ell_0} f_{j\ell}^{((p))} f_{kp}^{i} + \sum_{1 \leq p < j} f_{k\ell}^{((p))} f_{pj}^{i} - \sum_{j < p \leq \ell_0} f_{k\ell}^{((p))} f_{jp}^{i}$

ω	$\langle \omega, F(d_{2i-1}) \rangle \; (i = 1, \ldots, m)$
$z_j^* \wedge z_k^* \wedge d_{2i-1}^*$	$\sum_{p=1}^{k_0} c_{pi}^{(2i-1)} \, f_{jk}^{p} + \sum_{p=1}^{\ell_0} g_{pi}^{(2i-1)} \, f_{jk}^{((p))}$
$z_j^* \wedge z_k^* \wedge d_{2i}^*$	$\sum_{p=1}^{k_0} s_p^*(\Phi_i) \, f_{jk}^{p} + \sum_{p=1}^{\ell_0} \left(z_p^*(\Phi_i) + h_{pi}^{(2i-1)} \right) \, f_{jk}^{((p))}$
$d_{2j-1}^* \wedge d_{2j}^* \wedge d_{2i-1}^*$	$\sum_{p=1}^{k_0} c_{pi}^{(2i-1)} \, n_{jj}^{p} + \sum_{p=1}^{\ell_0} g_{pi}^{(2i-1)} \, n_{jj}^{((p))}$
$d_{2j-1}^* \wedge d_{2j}^* \wedge d_{2i}^*$	$\sum_{p=1}^{k_0} s_p^*(\Phi_i) \, n_{jj}^{p} + \sum_{p=1}^{\ell_0} \left(z_p^*(\Phi_i) + h_{pi}^{(2i-1)} \right) \, n_{jj}^{((p))} + d_{2j-1}^*(\Phi_i) d_{2j-1}^*(\Phi_j) + d_{2j}^*(\Phi_i) d_{2j}^*(\Phi_j)$

If the 1-cocycle $\xi = \operatorname{ad} r + R$ satisfies the co-Jacobi identity, i.e., $[\![\xi, \xi]\!] = 0$, then the condition $[\![R, R]\!] = 0$ is equivalent to the following system for all $i > k_0$:

$$\begin{cases} \displaystyle\sum_{p=1}^{\ell_0} h_{pi}^{(2i-1)} f_{jk}^{((p))} = 0, \\[2em] \displaystyle\sum_{p=1}^{\ell_0} h_{pi}^{(2i-1)} n_{jj}^{((p))} = 0, \end{cases} \tag{14}$$

where $f_{jk}^{((p))}$ and $n_{jj}^{((p))}$ encode the contribution of the central component $\mathfrak{z}$.

This condition is automatically satisfied in the following cases:

- When the center is trivial, i.e., $\mathfrak{z} = \{0\}$, any Lie bialgebra structure $(\mathfrak{g}, \operatorname{ad} r + R)$ implies that $(\mathfrak{g}, R)$ defines a non-exact Lie bialgebra.
- When $k_0 = m$, meaning the dimension of the derived ideal $[\mathfrak{g}, \mathfrak{g}]$ equals twice the dimension of the abelian subalgebra $\mathfrak{s}$, the vanishing of $[\![R, R]\!]$ is ensured when ξ satifies the co-Jacobi identity, and $(\mathfrak{g}, R)$ constitutes a non-exact Lie bialgebra structure.

ω	$\langle \omega, F(s_i) \rangle \; (i = 1, \ldots, k_0)$
$s_j^* \wedge s_k^* \wedge d_{2\ell-1}^*$	$\lambda_{i\ell} \left(-e_{k\ell} \, c_{j\ell}^{(2\ell-1)} + e_{j\ell} \, c_{k\ell}^{(2\ell-1)} - c_{k\ell} \, s_j^*(\Phi_\ell) + c_{j\ell} \, s_k^*(\Phi_\ell) \right)$
$s_j^* \wedge s_k^* \wedge d_{2\ell}^*$	$\lambda_{i\ell} \left(c_{k\ell} \, c_{j\ell}^{(2\ell-1)} - c_{j\ell} \, c_{k\ell}^{(2\ell-1)} - e_{k\ell} \, s_j^*(\Phi_\ell) + e_{j\ell} \, s_k^*(\Phi_\ell) \right)$
$s_j^* \wedge z_k^* \wedge d_{2\ell-1}^*$	$\lambda_{i\ell} \left(-h_{k\ell} \, c_{j\ell}^{(2\ell-1)} + e_{j\ell} \, g_{k\ell}^{(2\ell-1)} - g_{k\ell} \, s_j^*(\Phi_\ell) + c_{j\ell} \, (z_k^*(\Phi_\ell) + h_{k\ell}^{(2\ell-1)}) \right)$
$s_j^* \wedge z_k^* \wedge d_{2\ell}^*$	$\lambda_{i\ell} \left(g_{k\ell} \, c_{j\ell}^{(2\ell-1)} - c_{j\ell} \, g_{k\ell}^{(2\ell-1)} - h_{k\ell} \, s_j^*(\Phi_\ell) + e_{j\ell} \, (z_k^*(\Phi_\ell) + h_{k\ell}^{(2\ell-1)}) \right)$
$z_j^* \wedge z_k^* \wedge d_{2\ell-1}^*$	$-\lambda_{i\ell} \Big(\sum_{p=1}^{k_0} e_{p\ell} \, f_{jk}^{p} + \sum_{p=1}^{\ell_0} h_{p\ell} \, f_{jk}^{((p))} + h_{k\ell} g_{j\ell}^{(2\ell-1)} - h_{j\ell} g_{k\ell}^{(2\ell-1)}$ $+ g_{k\ell} \left(z_j^*(\Phi_\ell) + h_{j\ell}^{(2\ell-1)} \right) - g_{j\ell} \left(z_k^*(\Phi_\ell) + h_{k\ell}^{(2\ell-1)} \right) \Big)$
$z_j^* \wedge z_k^* \wedge d_{2\ell}^*$	$\lambda_{i\ell} \Big(\sum_{p=1}^{k_0} c_{p\ell} \, f_{jk}^{p} + \sum_{p=1}^{\ell_0} g_{p\ell} \, f_{jk}^{((p))} + g_{k\ell} g_{j\ell}^{(2\ell-1)} - g_{j\ell} g_{k\ell}^{(2\ell-1)}$ $- h_{k\ell} \left(z_j^*(\Phi_\ell) + h_{j\ell}^{(2\ell-1)} \right) + h_{j\ell} \left(z_k^*(\Phi_\ell) + h_{k\ell}^{(2\ell-1)} \right) \Big)$

ω	$\langle \omega, F(d_{2i-1})\rangle \; (i=1,\dots,m)$
$s_j^*\wedge d_{2i-1}^*\wedge d_{2i}^*$	$-c_{ji}\left(\sum_{p=1}^{k_0}\lambda_{pi}\, c_{pi}^{(2i-1)}\right) - s_j^*(\Phi_i)d_{2i}^*(\Phi_i) + c_{ji}^{(2i-1)}d_{2i-1}^*(\Phi_i)$
$z_j^*\wedge d_{2i-1}^*\wedge d_{2i}^*$	$-g_{ji}\left(\sum_{p=1}^{k_0}\lambda_{pi}\, c_{pi}^{(2i-1)}\right) - \left(z_j^*(\Phi_i)+h_{ji}^{(2i-1)}\right) d_{2i}^*(\Phi_i) + g_{ji}^{(2i-1)}d_{2i-1}^*(\Phi_i)$

ω	$\langle \omega, F(d_{2i})\rangle \; (i=1,\dots,m)$
$s_j^*\wedge d_{2i-1}^*\wedge d_{2i}^*$	$-e_{ji}\left(\sum_{p=1}^{k_0}\lambda_{pi}\, c_{pi}^{(2i-1)}\right) + s_j^*(\Phi_i)d_{2i-1}^*(\Phi_i) + c_{ji}^{(2i-1)}d_{2i}^*(\Phi_i)$
$z_j^*\wedge d_{2i-1}^*\wedge d_{2i}^*$	$-h_{ji}\left(\sum_{p=1}^{k_0}\lambda_{pi}\, c_{pi}^{(2i-1)}\right) + \left(z_j^*(\Phi_i)+h_{ji}^{(2i-1)}\right) d_{2i-1}^*(\Phi_i) + g_{ji}^{(2i-1)}d_{2i}^*(\Phi_i)$

The expressions for $F(d_{2i})$ are identical to those for $F(d_{2i-1})$ for all ω not listed in the table above, up to a sign change.

We will describe here all the possible solutions of the co-Jacobi equation.

We have for all $1\le j<k\le \ell_0$ and all $i=1\dots,m$

$$\begin{cases}\sum_{p=1}^{k_0} c_{pi}\, f_{jk}^{p} + \sum_{p=1}^{\ell_0} g_{pi}\, f_{jk}^{((p))} = 0\\ \sum_{p=1}^{k_0} e_{pi}\, f_{jk}^{p} + \sum_{p=1}^{\ell_0} h_{pi}\, f_{jk}^{((p))} = 0\end{cases}$$

For $i>k_0$ we have either

1. $$\begin{aligned} g_{ji}^{(2i-1)} &=0 = z_j^*(\phi_i)+h_{ji}^{(2i-1)}, && \text{for all } j=1,\dots,\ell_0\\ s_j^*(\phi_i) &=0, && \text{for all } j=1,\dots,k_0\\ d_{2j-1}^*(\phi_i)d_{2j-1}^*(\phi_j) &+ d_{2j}^*(\phi_i)d_{2j}^*(\phi_j)=0, && \text{for all } j=1,\dots,m,\ (j\neq i)\end{aligned}$$

   or

2. $d_{2i-1}^*(\phi_i)=0=d_{2i}^*(\phi_i)$ and

$$\begin{cases}\sum_{p=1}^{k_0} s_p^*(\phi_j)\, f_{ik}^{p} + \sum_{p=1}^{\ell_0}\left(z_p^*(\phi_j)+h_{pj}^{(2j-1)}\right)\, f_{ik}^{((p))}=0, & \text{for all } j=1,\dots,m\\ \sum_{p=1}^{k_0} s_p^*(\phi_j)\, n_{ii}^{p} + \sum_{p=1}^{\ell_0}\left(z_p^*(\phi_j)+h_{pj}^{(2j-1)}\right)\, n_{ii}^{((p))}=0, & \text{for all } j=1,\dots,m,\ (j\neq i)\\ \sum_{p=1}^{\ell_0} g_{pi}^{(2i-1)}\, f_{jk}^{((p))}=0, & \text{for all } 1\le j<k\le\ell_0\\ \sum_{p=1}^{\ell_0} g_{pi}^{(2i-1)}\, n_{jj}^{((p))}=0, & \text{for all } j=1,\dots,m\end{cases}$$

$$\begin{pmatrix}-c_{ki} & c_{ji}\\ -e_{ki} & e_{ji}\end{pmatrix}\begin{pmatrix}s_j^*(\phi_i)\\ s_k^*(\phi_i)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix},\ \text{for all } 1\le i<j\le k_0. \tag{15}$$

$$\begin{pmatrix}e_{ji} & c_{ji}\\ -c_{ji} & e_{ji}\end{pmatrix}\begin{pmatrix}g_{ki}^{(2i-1)}\\ z_k^*(\phi_i)+h_{ki}^{(2i-1)}\end{pmatrix} = s_j^*(\phi_i)\begin{pmatrix}g_{ki}\\ h_{ki}\end{pmatrix},$$
$$\text{for all } j=1,\dots,k_0,\ k=1,\dots,\ell_0. \tag{16}$$

We assume the condition (14) verified and we look for $r \in \bigwedge^2 \mathfrak{g}$ solution of the Yang-Baxter equation so that r defines a triangular Lie bialgebra structure compatible with the no exact one defined by R, i.e. $[\![\mathrm{ad}\, r, R]\!] = 0$.

3.3.2. Exact bialgebra structures

If $\xi = \mathrm{ad}\, r$ then, $(\mathfrak{g}, \xi)$ is a Lie bialgebra precisely when $[r, r]$ is a solution of the *Yang-Baxter equation*, that is,

$$[r, r] \in \left(\bigwedge^3 \mathfrak{g}\right)^{\mathfrak{g}}. \tag{17}$$

In particular, any bivector r satisfying $[r, r] = 0$ is called *triangular* (a solution to the classical Yang-Baxter equation).

The bracket we use is the *Schouten-Nijenhuis bracket*, defined for $X, Y \in \bigwedge^2 \mathfrak{g}$ and for all $\omega \in \bigwedge^3 \mathfrak{g}^*$ by Lichnerowicz's formula:

$$\begin{aligned}\langle \omega, [X, Y]\rangle &= -\langle d(i_Y\omega), X\rangle - \langle d(i_X\omega), Y\rangle + \langle d\omega, X \wedge Y\rangle \\ &= i_Y\omega(\mu(X)) + i_X\omega(\mu(Y)) + \langle d\omega, X \wedge Y\rangle,\end{aligned}$$

where the linear map $\mu : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$ is the Lie bracket of $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ be a nondegenerate flat Lie algebra. A cocycle $\xi = \mathrm{ad}\, r + R$ on $\mathfrak{g}$ is exact if and only if $R = 0$.

Since $r_0 \in (\mathfrak{s} \wedge [\mathfrak{g}, \mathfrak{g}]) \oplus (\mathfrak{z} \wedge [\mathfrak{g}, \mathfrak{g}]) \oplus \left(\underset{1\le i<j\le m}{\oplus} P_i \wedge P_j\right)$ and $r_1 \in \bigwedge^2 \mathfrak{s} \oplus \mathfrak{s} \wedge \mathfrak{z}$, then for $r = r_0 + r_1$, we have

$$[r, r] = [r_0, r_0] + 2[r_0, r_1] = m_1 + \ldots + m_8$$

corresponding to the direct-sum decomposition into $\mathfrak{s}$-modules noted respectively $M_1, \ldots, M_8$:

$$\begin{aligned}&\left(\oplus_{i=1}^m \mathfrak{s} \wedge \bigwedge^2 P_i\right) \oplus \left(\oplus_{i=1}^m \mathfrak{z} \wedge \bigwedge^2 P_i\right) \oplus \left(\underset{1\le i<j\le m}{\oplus} [\mathfrak{g}, \mathfrak{g}] \wedge P_i \wedge P_j\right) \\ &\qquad \oplus \left(\underset{1\le i<j\le m}{\oplus} \mathfrak{s} \wedge P_i \wedge P_j\right) \oplus \left(\underset{1\le i<j\le m}{\oplus} \mathfrak{z} \wedge P_i \wedge P_j\right) \\ &\qquad\qquad \oplus (\mathfrak{s} \wedge \mathfrak{z} \wedge [\mathfrak{g}, \mathfrak{g}]) \oplus \left(\bigwedge^2 \mathfrak{s} \wedge [\mathfrak{g}, \mathfrak{g}]\right) \oplus \left(\bigwedge^2 \mathfrak{z} \wedge [\mathfrak{g}, \mathfrak{g}]\right).\end{aligned}$$

If $[r, r]$ is $\mathfrak{g}$-invariant then $[r, r] = m_1 + m_2$ with $m_1 \in M_1^{\mathfrak{g}}$, $m_2 \in M_2^{\mathfrak{g}} = M_2$. This follows from the fact that $M_i^{\mathfrak{g}} = \{0\}$ for all $i = 3, \ldots, 8$. This relies on the following lemma:

Lemma 3.4. *Let* $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z} \oplus \left(\overset{m}{\underset{j=1}{\oplus}} P_i\right)$ *be a nondegenerate flat Lie algebra. Then*

1. $(P_i \wedge P_j \wedge P_k)^{\mathfrak{g}} = \{0\}$ *for all* $1 \le i < j < k \le m$.
2. $\left(\overset{m}{\underset{j=1}{\oplus}} \mathfrak{s} \wedge \bigwedge^2 P_j\right)^{\mathfrak{g}} = \left\{\sum_{i=1}^{k_0} \alpha_i\, s_i \wedge d_{2i-1} \wedge d_{2i}\right\}$, *where* $\alpha_i \neq 0$ *only if* $\lambda_{ij} = 0$ *for all* $j > k_0$.

Proof. 1. Suppose that

$$m = \Big(a_i\, d_{2i-1} + b_i\, d_{2i}\Big) \wedge \Big(a_j\, d_{2j-1} + b_j\, d_{2j}\Big) \wedge \Big(a_k\, d_{2k-1} + b_k\, d_{2k}\Big)$$

is an element of $P_i \wedge P_j \wedge P_k$. Then the condition

$$\mathrm{ad}_{s_p}\, m = 0 \quad \text{for all } p = 1, \ldots, k_0,$$

holds if and only if

$$\left(\begin{array}{cccc|cccc} 0 & -\lambda_{pk} & -\lambda_{pj} & 0 & -\lambda_{pi} & 0 & 0 & 0 \\ \lambda_{pk} & 0 & 0 & -\lambda_{pj} & 0 & -\lambda_{pi} & 0 & 0 \\ \lambda_{pj} & 0 & 0 & -\lambda_{pk} & 0 & 0 & -\lambda_{pi} & 0 \\ 0 & \lambda_{pj} & \lambda_{pk} & 0 & 0 & 0 & 0 & -\lambda_{pi} \\ \hline \lambda_{pi} & 0 & 0 & 0 & 0 & -\lambda_{pk} & -\lambda_{pj} & 0 \\ 0 & \lambda_{pi} & 0 & 0 & \lambda_{pk} & 0 & 0 & -\lambda_{pj} \\ 0 & 0 & \lambda_{pi} & 0 & \lambda_{pj} & 0 & 0 & -\lambda_{pk} \\ 0 & 0 & 0 & \lambda_{pi} & 0 & \lambda_{pj} & \lambda_{pk} & 0 \end{array}\right) \begin{pmatrix} a_i a_j a_k \\ a_i b_j a_k \\ a_i b_j b_k \\ b_i b_j b_k \\ b_i a_j a_k \\ b_i b_j a_k \\ b_i a_j b_k \\ a_i a_j b_k \end{pmatrix} = 0_{\mathbb{R}^8}.$$

This system can be rewritten in the block form

$$\left(\begin{array}{c|c} A & -\lambda_{pi}\,\mathrm{Id} \\ \hline \lambda_{pi}\,\mathrm{Id} & A \end{array}\right) \left(\begin{array}{c} X \\ \hline Y \end{array}\right) = 0_{\mathbb{R}^8}.$$

So $\lambda_{pi}\, Y = AX$ and $\big(A^2 + \lambda_{pi}^2\,\mathrm{Id}\big)\, X = 0$. Since the eigenvalues of A^2 are

$$-(\lambda_{pj} - \lambda_{pk})^2 \quad \text{and} \quad -\,(\lambda_{pj} + \lambda_{pk})^2,$$

and because $\mathfrak{g}$ is nondegenerate, there exists some $p \in \{1, \ldots, k_0\}$ for which $-\lambda_{pi}^2$ is not an eigenvalue of A^2. This forces $X = 0$ and hence $Y = 0$, implying that $m = 0$.

2. Assume that

$$m = s_i \wedge d_{2j-1} \wedge d_{2j}$$

is $\mathfrak{g}$-invariant. Then necessarily, we must have $i = j$; otherwise, one obtains $\mathrm{ad}_{d_{2i}}\, m = d_{2i-1} \wedge d_{2j-1} \wedge d_{2j} \neq 0$. Moreover, one computes

$$\mathrm{ad}_{d_{2p-1}}\, m = -\lambda_{ip}\, d_{2p} \wedge d_{2j-1} \wedge d_{2j}, \;\; \mathrm{ad}_{d_{2p}}\, m = \lambda_{ip}\, d_{2p-1} \wedge d_{2j-1} \wedge d_{2j}.$$

Thus, m is $\mathfrak{g}$-invariant if and only if $\lambda_{ip} = 0$ for all $p \neq i$, which is equivalent to saying that $\lambda_{ij} = 0$ for all $j > k_0$.

$\square$

ω	$\langle \omega\, , \, [r_0, r_1]\rangle$
$s_i^* \wedge z_j^* \wedge d_{2k-1}^*$	$-\left(\sum_{p<i}\lambda_{pk}(-a_{pi}) + \sum_{i<p}\lambda_{pk}a_{ip}\right) h_{jk}$
$s_i^* \wedge z_j^* \wedge d_{2k}^*$	$\left(\sum_{p<i}\lambda_{pk}(-a_{pi}) + \sum_{i<p}\lambda_{pk}a_{ip}\right) g_{jk}$
$z_i^* \wedge z_j^* \wedge d_{2k-1}^*$	$\left(\sum_{p=1}^{k_0}\lambda_{pk}b_{pi}\right) h_{jk} - \left(\sum_{p=1}^{k_0}\lambda_{pk}b_{pj}\right) h_{ik}$
$z_i^* \wedge z_j^* \wedge d_{2k}^*$	$-\left(\sum_{p=1}^{k_0}\lambda_{pk}b_{pi}\right) g_{jk} + \left(\sum_{p=1}^{k_0}\lambda_{pk}b_{pj}\right) g_{ik}$
$s_i^* \wedge d_{2j-1}^* \wedge d_{2k-1}^*$	$-\left(\sum_{p<i}\lambda_{pk}(-a_{pi}) + \sum_{i<p}\lambda_{pk}a_{ip}\right) n_{jk} + \left(\sum_{p<i}\lambda_{pj}(-a_{pi}) + \sum_{i<p}\lambda_{pj}a_{ip}\right) n_{kj}$
$s_i^* \wedge d_{2j-1}^* \wedge d_{2k}^*$	$\left(\sum_{p<i}\lambda_{pk}(-a_{pi}) + \sum_{i<p}\lambda_{pk}a_{ip}\right) m_{jk} - \left(\sum_{p<i}\lambda_{pj}(-a_{pi}) + \sum_{i<p}\lambda_{pj}a_{ip}\right) p_{jk}$
$s_i^* \wedge d_{2j}^* \wedge d_{2k}^*$	$\left(\sum_{p<i}\lambda_{pj}(-a_{pi}) + \sum_{i<p}\lambda_{pj}a_{ip}\right) n_{jk} - \left(\sum_{p<i}\lambda_{pk}(-a_{pi}) + \sum_{i<p}\lambda_{pk}a_{ip}\right) n_{kj}$
$z_i^* \wedge d_{2j-1}^* \wedge d_{2k-1}^*$	$\left(\sum_{p=1}^{k_0}\lambda_{pk}b_{pi}\right) n_{jk} - \left(\sum_{p=1}^{k_0}\lambda_{pj}b_{pi}\right) n_{kj}$
$z_i^* \wedge d_{2j-1}^* \wedge d_{2k}^*$	$-\left(\sum_{p=1}^{k_0}\lambda_{pk}b_{pi}\right) m_{jk} + \left(\sum_{p=1}^{k_0}\lambda_{pj}b_{pi}\right) p_{jk}$
$z_i^* \wedge d_{2j}^* \wedge d_{2k}^*$	$-\left(\sum_{p=1}^{k_0}\lambda_{pj}b_{pi}\right) n_{jk} + \left(\sum_{p=1}^{k_0}\lambda_{pk}b_{pi}\right) n_{kj}$

Theorem 3.5. *Let $(\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}], \operatorname{ad} r + R)$ a nondegenerate flat Lie algebra.*

1. *If $\mathfrak{g}^*$ is unimodular and $r \in \left(\bigwedge^2 \mathfrak{g}\right)^{\mathfrak{g}}$ then $[r, r] = 0$. More precisely, r_0 and r_1 are two compatible solutions of the classical Yang-Baxter equation.*
2. *If $r_1 = 0$ then the dual Lie algebra $\mathfrak{g}^*$ is not semisimple.*

Proof. 1. Recall that a Lie algebra is unimodular if its modular form vanishes identically. For the dual Lie algebra $\mathfrak{g}^*$, this condition can be expressed as follows:

$$\operatorname{tr}(\operatorname{ad}^*_{s_i}) = 2c_{ii}^{(2i-1)}, \quad \operatorname{tr}(\operatorname{ad}^*_{d_{2i-1}}) = 2d_{2i}^*(\Phi_i), \quad \operatorname{tr}(\operatorname{ad}^*_{d_{2i}}) = -2d_{2i-1}^*(\Phi_i),$$

and

$$\operatorname{tr}(\operatorname{ad}^*_{z_i}) = 2\sum_{p=1}^{m} g_{ip}^{(2p-1)} - \sum_{p<i} f_{pi}^{(p)} + \sum_{i<p} f_{ip}^{(p)}.$$

As established earlier, the condition $[r, r] \in \left(\bigwedge^3 \mathfrak{g}\right)^{\mathfrak{g}}$ (that is, the Yang-Baxter equation) is satisfied if and only if $r = r_0 + r_1$ lies in the subspace

$$\bigoplus_{j=1}^{m}\left(\mathfrak{s}\wedge\bigwedge^2 P_j\right)^{\mathfrak{s}}\oplus\bigoplus_{j=1}^{m}\left(\mathfrak{z}\wedge\bigwedge^2 P_j\right),$$

where r_0 and r_1 are as described in the main theorem. In particular, this implies that $\mu(r) = \mu(r_0) = 0$, where μ denotes the Lie bracket. Moreover, for any 3-form ω in $\mathfrak{s}^*\wedge\bigwedge^2 P_j^*\oplus\mathfrak{z}^*\wedge\bigwedge^2 P_j^*$, we have $d\omega = 0$, so

$$\langle[r,r],\omega\rangle = 2\langle r\wedge\mu(r),\omega\rangle + \langle r\wedge r, d\omega\rangle = 0.$$

Similarly, $\langle[r_0,r_0],\omega\rangle = 0$, and thus $[r,r] = [r_0,r_0] = 0$. Therefore, both r_0 and r_1 provide compatible solutions to the classical Yang-Baxter equation.

2. For the second statement, suppose $r_1 = 0$ and assume, for contradiction, that the dual Lie algebra $\mathfrak{g}^*$ is semisimple, thus unimodular. Observe that, for all $i = 1,\dots,k_0$ and $j = 1,\dots,m$, the dual Lie brackets are given by

$$[s_i^*, d_{2j-1}^*] = -e_{ij}\sum_{p=1}^{k_0}\lambda_{pj}s_p^*,\qquad [s_i^*, d_{2j}^*] = c_{ij}\sum_{p=1}^{k_0}\lambda_{pj}s_p^*.$$

This demonstrates that s^* is a nontrivial abelian ideal in $\mathfrak{g}^*$, which is a contradiction. Therefore, $\mathfrak{g}^*$ cannot be semisimple if $r_1 = 0$.

□

4. Examples

4.1. Low dimensional examples

Dimension 3. Let $\mathfrak{g} = \mathrm{span}\{s, d_1, d_2\}$ be the flat Lie algebra with the brackets $[s,d_1] = d_2$, $[s,d_2] = -d_1$. A 1-cocycle on $\mathfrak{g}$ is of the form

$$\begin{cases} \xi(s) &= a\,s\wedge d_1 + b\,s\wedge d_2 + c\,d_1\wedge d_2\\ \xi(d_1) &= e\,s\wedge d_1 + b\,d_1\wedge d_2\\ \xi(d_2) &= e\,s\wedge d_2 - a\,d_1\wedge d_2.\end{cases}$$

The transpose map ξ^t is a Lie bracket on the dual vector space $\mathfrak{g}^*$ if and only if $c = 0$ or $e = 0$. In both cases the dual Lie algebra is solvable.

For any $r\in\bigwedge^2\mathfrak{g}$, $[r,r]$ is ad-invariant and

$$[r,r] = 0 \Leftrightarrow r = c\,d_1\wedge d_2\ (c\in\mathbb{R}),$$

i.e. the unique solution to the classical Yang-Baxter equation consists of the invariant elements within $\bigwedge^2\mathfrak{g}$ which give rise to a trivial cocycle.

Dimension 4. Let $\mathfrak{g} = \operatorname{span}\{s, z, d_1, d_2\}$ be the flat Lie algebra with the brackets $[s, d_1] = d_2$, $[s, d_2] = -d_1$. A 1-cocycle on $\mathfrak{g}$ is of the form

$$\begin{cases} \xi(z) &= & n_{11}^{((1))}\, d_1 \wedge d_2 \\ \xi(s) &= & c_{11}\, s \wedge d_1 + e_{11}\, s \wedge d_2 + g_{11}\, z \wedge d_1 + h_{11}\, z \wedge d_2 + n_{11}\, d_1 \wedge d_2 \\ \xi(d_1) &= & C_{11}^{(1)}\, s \wedge d_1 + g_{11}^{(1)} z\, d_1 + h_{11}^{(1)} z\, d_2 + e_{11}\, d_1 \wedge d_2 \\ \xi(d_2) &= & C_{11}^{(1)}\, s \wedge d_2 - h_{11}^{(1)} z\, d_1 + g_{11}^{(1)} z\, d_2 - c_{11}\, d_1 \wedge d_2. \end{cases}$$

The transpose map ξ^t is a Lie bracket on the dual vector space $\mathfrak{g}^*$ if and only if

$$\begin{cases} n_{11}^{((1))} = n_{11} = 0 & \text{or} \quad c_{11} = 0 \\ g_{11}c_{11}^{(1)} - c_{11}g_{11}^{(1)} + e_{11}h_{11}^{(1)} &= \quad 0 \\ h_{11}c_{11}^{(1)} - e_{11}g_{11}^{(1)} - c_{11}h_{11}^{(1)} &= \quad 0. \end{cases}$$

In these three cases the dual Lie algebra is solvable.

For any $r \in \bigwedge^2 \mathfrak{g}$, $[r, r]$ is ad-invariant and

$$[r, r] = 0 \Leftrightarrow r = a\, s \wedge z + b\, z \wedge d_1 + c\, z \wedge d_2 \ (a, b, c \in \mathbb{R}).$$

5. Flat Poisson-Lie groups

The simply connected Lie group whose Lie algebra is flat is given by $G = \mathbb{R}^{k_0} \ltimes \mathbb{R}^{\ell_0 + 2m}$ with the product $(x, y, z) \cdot (X, Y, z) := (x + X, y + Y, z + \operatorname{ad}_x Z)$, where

$$\operatorname{ad}_x Z = e^{\operatorname{ad}_x} Z = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_{2m} \end{pmatrix}$$

where each J_ℓ is a 2×2 skew-symmetric matrix.

The spaces of left invariant and right invariant vector fields are given respectively by:

$$\begin{aligned} \mathcal{X}_L(G) =& \langle \partial_{x_1}, \dots, \partial_{x_{k_0}} \rangle \oplus \langle \partial_{y_1}, \dots, \partial_{y_{\ell_0}} \rangle \oplus \langle D_1, \dots, D_{2m} \rangle \\ \mathcal{X}_R(G) =& \langle E_1, \dots, E_{k_0} \rangle \oplus \langle \partial_{y_1}, \dots, \partial_{y_{\ell_0}} \rangle \oplus \langle \partial_{z_1}, \dots, \partial_{z_{2m}} \rangle, \end{aligned}$$

where

$$\begin{cases} D_{2\ell-1} = & \cos\left(\sum_{i=1}^{k_0} \lambda_{i\ell} x_i\right) \partial_{z_{2\ell-1}} + \sin\left(\sum_{i=1}^{k_0} \lambda_{i\ell} x_i\right) \partial_{z_{2\ell}} \\ D_{2\ell} \quad = & -\sin\left(\sum_{i=1}^{k_0} \lambda_{i\ell} x_i\right) \partial_{z_{2\ell-1}} + \cos\left(\sum_{i=1}^{k_0} \lambda_{i\ell} x_i\right) \partial_{z_{2\ell}} \end{cases}$$

and

$$E_k = \partial_{x_k} + \lambda_{k1}(-z_2 \partial_{z_1} + z_1 \partial_{z_2}) + \ldots + \lambda_{km}(-z_{2m} \partial_{z_{2m-1}} + z_{2m-1} \partial_{z_{2m}})$$

6. Conclusion

Despite the apparent structural simplicity of flat Lie algebras, the comprehensive classification of bialgebra structures on these algebras presents significant computational challenges., with the degenerate case remaining unexplored. Given that degenerate flat Lie algebras can be obtained as contractions of their nondegenerate counterparts, this suggests that insights from this work may extend to the degenerate setting.

Acknowledgement

The author wishes to thank the editor and the referee for their valuable comments which improved the presentation of the paper.

Funding. This research received no external funding.

Data Availability Statement. Not applicable.

Competing interests. The author declares no competing interests.

Supplementary Materials. None.

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