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Research Article

An Existence Result for a Generalized Quasilinear Schrödinger Equation with Nonlocal Term

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In this paper, we consider the following generalized quasilinear Schrödinger equation with nonlocal term $-\mathrm{div}\,(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = \lambda[|x|^{-\mu} * |u|^p]|u|^{p-2}u, x \in \mathbb{R}^N$, where $N \geq 3$, $g : \mathbb{R} \to \mathbb{R}^+$ is a C^1 even function, $g(0) = 1$, $g'(s) \geq 0$ is for all $s \geq 0$, $\lim_{|s| \to +\infty} g(s)/|s|^{\alpha-1} := \beta > 0$ is for some $\alpha > 1$, and $(\alpha - 1)g(s) \geq g'(s)s$ is for all $s \geq 0$, $2\alpha \leq p \leq 2\alpha(N - \mu)/N - 2$, and $0 < \mu < N$. We prove that the equation admits a solution by using a constrained minimization argument.

1. Introduction and Preliminaries

The main purpose of this paper is to investigate the existence of solutions for the following generalized quasilinear Schrödinger equation with nonlocal term

$$-\mathrm{div}\,(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = \lambda\left[|x|^{-\mu} * |u|^p\right]|u|^{p-2}u, x \in \mathbb{R}^N, \tag{1}$$

where $N \geq 3$, $g : \mathbb{R} \to \mathbb{R}^+$ is a C^1 even function, $g(0) = 1$, $g'(s) \geq 0$ is for all $s \geq 0$, $\lim_{|s| \to +\infty} g(s)/|s|^{\alpha-1} := \beta > 0$ is for some $\alpha > 1$, and $(\alpha - 1)g(s) \geq g'(s)s$ is for all $s \geq 0$, $2\alpha \leq p \leq 2\alpha(N - \mu)/N - 2$, and $0 < \mu < N$.

When $g(u) = 1$, (1) boils down to the socalled nonlinear Choquard or Choquard-Pekar equation

$$-\Delta u + V(x)u = \lambda\left[|x|^{-\mu} * |u|^p\right]|u|^{p-2}u, x \in \mathbb{R}^N. \tag{2}$$

Such like equation has several physical origins. The problem

$$-\Delta u + u = \left[|x|^{-1} * |u|^2\right]u, x \in \mathbb{R}^3 \tag{3}$$

appeared at least as early as in 1954, in a work by Pekar describing the quantum mechanics of a polaron at rest [1]. In 1976, Choquard used (3) to describe an electron trapped in its own hole and in a certain approximation to Hartree-Fock theory of one component plasma [2]. In 1996, Penrose proposed (3) as a model of self-gravitating matter, in a program in which quantum state reduction is understood as a gravitational phenomenon [3]. In this context, equation of type (3) is usually called the nonlinear Schrödinger-Newton equation. The first investigations for existence and symmetry of the solutions to (3) go back to the works of Lieb [2] and Lions [4]. In [2], by using symmetric decreasing rearrangement inequalities, Lieb proved that the ground state solution of equation (3) is radial and unique up to translations. Lions

[4] showed the existence of a sequence of radially symmetric solutions. Since then, many efforts have been made to study the existence of nontrivial solutions for nonlinear Choquard equations. Wei and Winter [5] showed that the ground state solution is nondegenerate. Ma and Zhao [6] considered the generalized Choquard equation

$$-\Delta u+u=[|x|^{-\mu}*|u|^{q}]|u|^{q-2}u(q\geq 2) \tag{4}$$

and proved that every positive solution of it is radially symmetric and monotone decreasing about some fixed point, under the assumption that a certain set of real numbers, defined in terms of N, μ, and q, is nonempty. Under the same assumption, Cingolani, Clapp, and Secchi [7] gave some existence and multiplicity results in the electromagnetic case and established the regularity and some decay asymptotically at infinity of the ground states. In [8], Moroz and Van Schaftingen eliminated this restriction and showed the regularity, positivity, and radial symmetry of the ground states for the optimal range of parameters and derived decay asymptotically at infinity for them as well. Moreover, they [9] also obtained a similar conclusion under the assumption of Berestycki-Lions type nonlinearity. We point out that the existence, multiplicity, and concentration of such like equation have been established by many authors. We refer the readers to [10, 11] for the existence of sign-changing solutions, [5, 12] for the existence and concentration behavior of the semiclassical solutions and [13] for the critical nonlocal part with respect to the Hardy-Littlewood-Sobolev inequality. For more details associated with the Choquard equation, please refer to [14–16] and the references therein.

In the past, even the research on the existence of solitary wave solutions for the Schrödinger equation with local term

$$-\text{div}\left(g^2(u)\nabla u\right)+g(u)g'(u)|\nabla u|^2+V(x)u=f(x,u), x\in\mathbb{R}^N \tag{5}$$

is for some given special function $g(\cdot)$, see [17–19]. However, related to the nonlocal equation (1), as far as we know, there is no result in this direction. In this paper, with the aid of the new variable replacement developed by Shen and Wang in [18] and inspired by [20, 21], existence of solutions for equation (1) have been established. Problem (1) has a variational structure, and the corresponding energy functional is defined by

$$\begin{aligned}I(u)&=\frac{1}{2}\int_{\mathbb{R}^N}g^2(u)|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx\\&\quad-\frac{\lambda}{2p}\int_{\mathbb{R}^N}\left[|x|^{-\mu}*|u|^p\right]|u|^pdx\\&=\frac{1}{2}\int_{\mathbb{R}^N}g^2(u)|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx\\&\quad-\frac{\lambda}{2p}\int_{\mathbb{R}^{2N}}\frac{|u(x)|^p|u(y)|^p}{|x-y|^\mu}dxdy.\end{aligned} \tag{6}$$

However, I is not well defined in $H^1(\mathbb{R}^N)$ because of the term $\int_{\mathbb{R}^N}g^2(u)|\nabla u|^2dx$. To overcome this difficulty, we make a change of variable constructed by Shen and Wang in [18]: $v\coloneqq G(u)\coloneqq\int_0^u g(t)dt$. Then, we obtain

$$\begin{aligned}J(v)&=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)G^{-1}(v)^2dx\\&\quad-\frac{\lambda}{2p}\int_{\mathbb{R}^N}\left[|x|^{-\mu}*\left|G^{-1}(v)\right|^p\right]\left|G^{-1}(v)\right|^pdx\\&=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)G^{-1}(v)^2dx\\&\quad-\frac{\lambda}{2p}\int_{\mathbb{R}^{2N}}\frac{\left|G^{-1}(v(x))\right|^p\left|G^{-1}(v(y))\right|^p}{|x-y|^\mu}dxdy.\end{aligned} \tag{7}$$

We say that u is a weak solution of (1), if

$$\begin{aligned}\left\langle I'(u),\varphi\right\rangle&=\int_{\mathbb{R}^N}\left\{g^2(u)\nabla u\nabla\varphi+g(u)g'(u)|\nabla u|^2\varphi\right.\\&\quad\left.+V(x)u\varphi-\lambda\left[|x|^{-\mu}*|u|^p\right]|u|^{p-2}u\varphi\right\}dx=0\end{aligned} \tag{8}$$

for all $\varphi\in C_0^\infty(\mathbb{R}^N)$. Let $\varphi=(1/g(u))\psi$. By [18], we know that the above formula is equivalent to

$$\begin{aligned}\left\langle J'(v),\psi\right\rangle&=\int_{\mathbb{R}^N}\left\{\nabla v\nabla\psi+V(x)\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}\psi\right.\\&\quad\left.-\lambda\frac{\left[|x|^{-\mu}*\left|G^{-1}(v)\right|^p\right]\left|G^{-1}(v)\right|^{p-2}G^{-1}(v)\psi}{g\left(G^{-1}(v)\right)}\right\}dx=0\end{aligned} \tag{9}$$

for all $\psi\in C_0^\infty(\mathbb{R}^N)$. Therefore, in order to find the solution of (1), it suffices to study the solution of the following equation:

$$-\Delta v+V(x)\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}-\lambda\frac{\left[|x|^{-\mu}*\left|G^{-1}(v)\right|^p\right]\left|G^{-1}(v)\right|^{p-2}G^{-1}(v)}{g\left(G^{-1}(v)\right)}=0. \tag{10}$$

In this paper, we assume that the following condition holds.

(V) $V\in C(\mathbb{R}^N,\mathbb{R})$, $0<V_0\coloneqq\inf_{x\in\mathbb{R}^N}V(x)$, and $\lim_{|x|\to\infty}V(x)=+\infty$.

Set $H_V^1(\mathbb{R}^N)=\{v\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N}[|\nabla v|^2+V(x)v^2]dx<+\infty\}$ with the norm

$$\|v\|_{H_V^1}^2=\int_{\mathbb{R}^N}\left[|\nabla v|^2+V(x)v^2\right]dx. \tag{11}$$

Then, by the proof of Lemma 4 in [22], the embedding

$H_V^1(\mathbb{R}^N)^\circ L^t(\mathbb{R}^N)$ is compact for all $t \in [2, 2^*)$. Moreover, for any $a > 0$, we define $m_a := \inf_{v \in M_a} E(v)$, where

$$M_a := \left\{ v \in H_V^1(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|G^{-1}(v(x))|^p |G^{-1}(v(y))|^p}{|x-y|^\mu} dxdy = a \right\}$$
$$E(v) := \int_{\mathbb{R}^N} \left[|\nabla v|^2 + V(x) G^{-1}(v)^2 \right] dx. \tag{12}$$

Our main result is the following:

Theorem 1. *Suppose that* (V) *is satisfied, then, there exists* $\lambda_n \to +\infty$ *such that equation (1) with* $\lambda = \lambda_n$ *has a solution.*

2. Proof of Theorem 1

To begin with, we give some lemmas.

Lemma 2 (see [23, 24]). *The functions* g, G *and* G^{-1} *possess the following properties:*

(1) $G(s) \le g(s)s \le \alpha G(s)$ *for all* $s \ge 0$; $G(s) \ge g(s)s \ge \alpha G(s)$ *for all* $s \le 0$

(2) $G^{-1}(s)s/g(G^{-1}(s)) \le |G^{-1}(s)|^2 \le \alpha (G^{-1}(s)s/g(G^{-1}(s)))$ *for all* $s \in \mathbb{R}$

(3) $|s|^\alpha \le (\alpha/\beta) \mid G(s) \mid$ *for all* $s \in \mathbb{R}$

Proposition 3 [25] (Hardy-Littlewood-Sobolev inequality). *Let* $r, t > 1$ *and* $0 < \mu < N$ *with* $(1/r) + (\mu/N) + (1/t) = 2$. *Let* $g \in L^r(\mathbb{R}^N)$ *and* $h \in L^t(\mathbb{R}^N)$. *Then, there exists a sharp constant* $C_{r,N,\mu,t}$ *independent of* g *and* h *such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x-y|^\mu} dxdy \right| \le C_{r,N,\mu,t} \|g\|_r \|h\|_t. \tag{13}$$

Proof of Theorem 1. The proof consists of two steps.

Step 1: we prove that for each $a > 0$, m_a is achieved at some $v_a \in M_a$, which is a weak solution of equation (10) with $\lambda = \lambda_a$ satisfying $\lambda_a \in [(m_a/a\alpha), (\alpha m_a/a)]$.

For fixed $a > 0$, let $\{v_n\} \subset M_a$ be a minimizing sequence for m_a, i.e., $v_n \in H_V^1(\mathbb{R}^N)$ satisfying $\int_{\mathbb{R}^{2N}} (|G^{-1}(v_n(x))|^p |G^{-1}(v_n(y))|^p / |x-y|^\mu) dxdy = a$ such that $E(v_n) \to m_a$ as $n \to \infty$. We assert that there exists a constants $C_1 > 0$ such that $E(v_n) \ge C_1 \int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x) v_n^2] dx$. Indeed, we may assume that $v_n \ne 0$ (otherwise, the conclusion is trivial). If the conclusion is false, then for any positive integer n, we may assume that

$$E(v_n) = \int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x) G^{-1}(v_n)^2 \right] dx < \frac{1}{n} \|v_n\|^2_{H_V^1} = \frac{1}{n} \int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x) v_n^2 \right] dx. \tag{14}$$

Set $w_n = v_n / \|v_n\|_{H_V^1}$ and $g_n = G^{-1}(v_n)^2 / \|v_n\|^2_{H_V^1}$. Then,

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \int_{\mathbb{R}^N} V(x) g_n(x) dx \to 0, \tag{15}$$

which implies that

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 dx \to 0, \int_{\mathbb{R}^N} V(x) g_n(x) dx \to 0, \int_{\mathbb{R}^N} V(x) w_n^2 dx \to 1 \tag{16}$$

as $n \to \infty$. Then for each $\varepsilon > 0$, there exists a constant $C_2 > 0$ independent of n such that $meas(\Omega_n) < \varepsilon$, where $\Omega_n := \{x \in \mathbb{R}^N : |G^{-1}(v_n(x))| \ge C_2\}$. Otherwise, there exist $\varepsilon_0 > 0$ and a subsequence $\{G^{-1}(v_{n_k})\}$ of $\{G^{-1}(v_n)\}$ such that for any positive integer k,

$$meas(\Omega_{n_k}) \ge \varepsilon_0 > 0, \tag{17}$$

where $\Omega_{n_k} = \{x \in \mathbb{R}^N : |G^{-1}(v_{n_k}(x))| \ge k\}$. By (V), one has

$$C \ge E(v_{n_k}) \ge \int_{\mathbb{R}^N} V(x) G^{-1}(v_{n_k}(x))^2 dx \ge \int_{\Omega_{n_k}} V(x) G^{-1}(v_{n_k}(x))^2 dx \ge V_0 k^2 \varepsilon_0 \to +\infty \tag{18}$$

as $k \to +\infty$, a contradiction. Noting that as $|G^{-1}(v_n)| < C_2$, by Lemma 2 (1) and monotonicity of g, we have

$$v_n^2 \le g^2(G^{-1}(v_n)) G^{-1}(v_n)^2 \le g^2(C_2) G^{-1}(v_n)^2. \tag{19}$$

Hence,

$$\int_{\mathbb{R}^N \setminus \Omega_n} V(x) w_n^2 dx \le g^2(C_2) \int_{\mathbb{R}^N} V(x) g_n(x) dx \to 0. \tag{20}$$

By the integral absolutely continuity, there exists $\varepsilon > 0$ such that whenever $\Omega \subset \mathbb{R}^N$ and $meas(\Omega) < \varepsilon$, $\int_\Omega V(x) w_n^2 dx < 1/2$. For this ε, one has

$$\int_{\mathbb{R}^N} V(x) w_n^2 dx = \int_{\Omega_n} V(x) w_n^2 dx + \int_{\mathbb{R}^N \setminus \Omega_n} V(x) w_n^2 dx \le \frac{1}{2} + \int_{\mathbb{R}^N \Omega_n} V(x) w_n^2 dx, \tag{21}$$

which implies $1 \le 1/2$, a contradiction. Therefore, up to a subsequence, there exists $v_a \in H_V^1(\mathbb{R}^N)$ such that $v_n \to v_a$ in $H_V^1(\mathbb{R}^N)$, $v_n \to v_a$ in $L^t(\mathbb{R}^N)$ for $2 \le t < 2^*$, and $v_n(x) \to v_a(x)$, a.e., on $\mathbb{R}^N$. By means of the definition of weak convergence, we know

$$0 \leq \int_{\mathbb{R}^N} |\nabla v_n - \nabla v_a|^2 dx = \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} |\nabla v_a|^2 dx - 2\int_{\mathbb{R}^N} \nabla v_n \cdot \nabla v_a dx, \tag{22}$$

which implies that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \geq \liminf_{n\to\infty} \left[2\int_{\mathbb{R}^N} \nabla v_n \cdot \nabla v_a dx - \int_{\mathbb{R}^N} |\nabla v_a|^2 dx\right] = \int_{\mathbb{R}^N} |\nabla v_a|^2 dx. \tag{23}$$

By Fatou Lemma, we have

$$\int_{\mathbb{R}^N} V(x) G^{-1}(v_a)^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x) G^{-1}(v_n)^2 dx. \tag{24}$$

Consequently, $E(v_a) \leq \liminf_{n\to\infty} [\int_{\mathbb{R}^N} |\nabla v_n|^2 + \int_{\mathbb{R}^N} V(x) G^{-1}(v_n)^2] = \liminf_{n\to\infty} E(v_n)$. Moreover, by the Hardy-Littlewood-Sobolev inequality and Lemma 2 (3), one has

$$a = \int_{\mathbb{R}^{2N}} \frac{|G^{-1}(v_n(x))|^p |G^{-1}(v_n(y))|^p}{|x-y|^\mu} dxdy \leq C \| |G^{-1}(v_n)|^p \|^2_{2N/2N-\mu} \leq C\left(\int_{\mathbb{R}^N} |v_n|^{2Np/\alpha(2N-\mu)} dx\right)^{2N-\mu/N}. \tag{25}$$

Since $2 < 2Np/\alpha(2N-\mu) < 2^*$, by Lemma A.1 in [26] and Lebesgue's dominated convergence theorem, we can easily infer that $\int_{\mathbb{R}^{2N}} (|G^{-1}(v_a(x))|^p |G^{-1}(v_a(y))|^p / |x-y|^\mu) dxdy = a$, and so $v_a \in M_a$. Hence, $m_a \leq E(v_a) \leq \liminf_{n\to\infty} E(v_n) = m_a$, which means that m_a is achieved at some $v_a \in M_a$. Moreover, by a standard argument, we can conclude that v_a is a weak solution of

$$-\Delta v + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} = \lambda_a \frac{\left[|x|^{-\mu} * |G^{-1}(v)|^p\right] |G^{-1}(v)|^{p-2} G^{-1}(v)}{g(G^{-1}(v))}. \tag{26}$$

Multiplying the above equation by v_a and integrating over $\mathbb{R}^N$, one has

$$\int_{\mathbb{R}^N} |\nabla v_a|^2 dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v_a) v_a}{g(G^{-1}(v_a))} dx = \lambda_a \int_{\mathbb{R}^N} \frac{\left[|x|^{-\mu} * |G^{-1}(v_a)|^p\right] |G^{-1}(v_a)|^{p-2} G^{-1}(v_a) v_a}{g(G^{-1}(v_a))} dx. \tag{27}$$

By Lemma 2 (2), we obtain $m_a/a\alpha \leq \lambda_a \leq \alpha m_a/a$. Indeed, by Lemma 2 (2), we have

$$\begin{aligned} \frac{1}{\alpha} m_a = \frac{1}{\alpha} E(v_a) &= \frac{1}{\alpha}\left[\int_{\mathbb{R}^N} |\nabla v_a|^2 dx + \int_{\mathbb{R}^N} V(x) G^{-1}(v_a)^2 dx\right] \\ &\leq \int_{\mathbb{R}^N} |\nabla v_a|^2 dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} V(x) G^{-1}(v_a)^2 dx \\ &\leq \int_{\mathbb{R}^N} |\nabla v_a|^2 dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v_a) v_a}{g(G^{-1}(v_a))} dx \\ &\leq \lambda_a \int_{\mathbb{R}^N} \left[|x|^{-\mu} * |G^{-1}(v_a)|^p\right] |G^{-1}(v_a)|^p dx = \lambda_a \cdot a, \end{aligned} \tag{28}$$

i.e., $\lambda_a \geq m_a/a\alpha$. Furthermore,

$$\begin{aligned} m_a = E(v_a) &= \int_{\mathbb{R}^N} |\nabla v_a|^2 dx + \int_{\mathbb{R}^N} V(x) G^{-1}(v_a)^2 dx \\ &\geq \int_{\mathbb{R}^N} |\nabla v_a|^2 dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v_a) v_a}{g(G^{-1}(v_a))} dx \\ &\geq \frac{\lambda_a}{\alpha} \int_{\mathbb{R}^N} \left[|x|^{-\mu} * |G^{-1}(v_a)|^p\right] |G^{-1}(v_a)|^p dx = \frac{\lambda_a}{\alpha} \cdot a, \end{aligned} \tag{29}$$

i.e., $\lambda_a \leq \alpha m_a/a$.

Step 2: we prove that $\lambda_a \to +\infty$ as $a \to 0$.

If the conclusion is false, then there exists a constant $G_0 > 0$ and $a_n \to 0 (n \to \infty)$ such that $\lambda_n := \lambda_{a_n} \leq G_0$. Set $v_n := v_{a_n}$, by Lemma 2 (2) and Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v_n) v_n}{g(G^{-1}(v_n))} dx \\ &= \lambda_n \int_{\mathbb{R}^N} \frac{\left[|x|^{-\mu} * |G^{-1}(v_n)|^p\right] |G^{-1}(v_n)|^{p-2} G^{-1}(v_n) v_n}{g(G^{-1}(v_n))} dx \\ &\leq \lambda_n \int_{\mathbb{R}^N} \left[|x|^{-\mu} * |G^{-1}(v_n)|^p\right] |G^{-1}(v_n)|^p dx = \lambda_n a_n \\ &\leq G_0 \cdot a_n \to 0 \end{aligned} \tag{30}$$

as $n \to \infty$. Since $2 < 2Np/\alpha(2N-\mu) < 2^*$, there exists a constant $\theta \in (0,1)$ such that $1/2Np/\alpha(2N-\mu) = (\theta/2) + (1-\theta/2^*)$. Consequently, by Lemma 2 (3), (V), Hölder inequality, and Young inequality, one has

$$\begin{aligned}\int_{\mathbb{R}^N}\left|G^{-1}(v_n)\right|^{2Np/2N-\mu}dx &\le C\int_{\mathbb{R}^N}|v_n|^{2Np/\alpha(2N-\mu)}dx\\ &= C\int_{\mathbb{R}^N}|v_n|^{\theta 2Np/\alpha(2N-\mu)}|v_n|^{(1-\theta)2Np/\alpha(2N-\mu)}dx\\ &\le C\|v_n\|_2^{\theta 2Np/\alpha(2N-\mu)}\|v_n\|_{2^*}^{(1-\theta)2Np/\alpha(2N-\mu)}\\ &\le C\theta\|v_n\|_2^{2Np/\alpha(2N-\mu)}\\ &\quad + C(1-\theta)\|v_n\|_{2^*}^{2Np/\alpha(2N-\mu)}\\ &\le C\left(\int_{\mathbb{R}^N}V(x)v_n^2dx\right)^{Np/\alpha(2N-\mu)}\\ &\quad + C\left(\int_{\mathbb{R}^N}|\nabla v_n|^2dx\right)^{Np/\alpha(2N-\mu)}\\ &\le C\left(\int_{\mathbb{R}^N}\left[|\nabla v_n|^2+V(x)v_n^2\right]dx\right)^{Np/\alpha(2N-\mu)}\\ &\le C\left(\int_{\mathbb{R}^N}\left[|\nabla v_n|^2+V(x)G^{-1}(v_n)^2\right]dx\right)^{Np/\alpha(2N-\mu)}.\end{aligned} \tag{31}$$

Hence, again, by Lemma 2 (2) and Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla v_n|^2dx+\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))}dx\\ &\quad\le \lambda_n\int_{\mathbb{R}^N}\left[|x|^{-\mu}*\left|G^{-1}(v_n)\right|^p\right]\left|G^{-1}(v_n)\right|^pdx\\ &\quad\le C\lambda_n\left\|\left|G^{-1}(v_n)\right|^p\right\|_{2N/2N-\mu}^2\\ &\quad\le C\cdot G_0\left(\int_{\mathbb{R}^N}\left[|\nabla v_n|^2+V(x)G^{-1}(v_n)^2\right]dx\right)^{p/\alpha}\\ &\quad\le C\cdot G_0\left(\int_{\mathbb{R}^N}\left[|\nabla v_n|^2+V(x)\frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))}\right]dx\right)^{p/\alpha},\end{aligned} \tag{32}$$

and so $\int_{\mathbb{R}^N}[|\nabla v_n|^2+V(x)(G^{-1}(v_n)v_n/g(G^{-1}(v_n)))]dx\ge C$ since $p/\alpha\ge 2>1$, a contradiction. By steps 1 and 2, we complete the proof of Theorem 1.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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